Lattice quotients of weak order intervals in subword complexes

Noémie Cartier

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Joint work with :
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Cesar Ceballos
Vincent Pilaud

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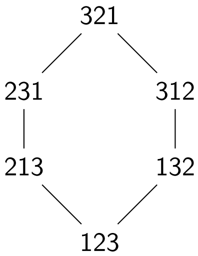

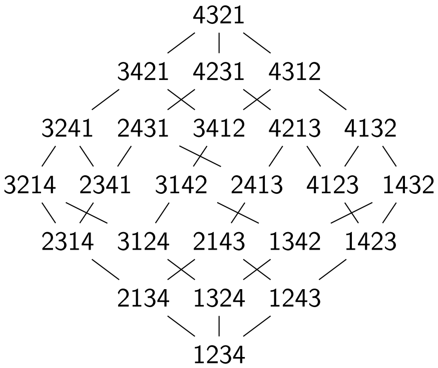

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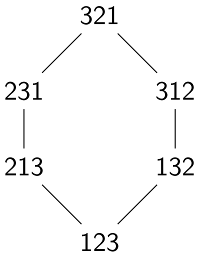

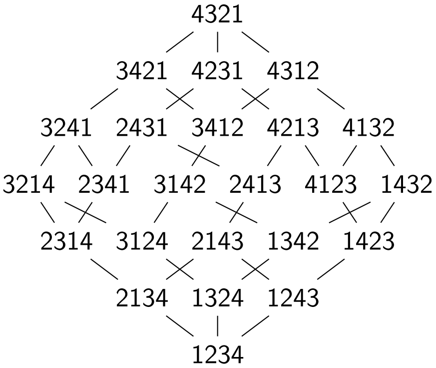

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Theorem

*The weak order on $\mathfrak{S}_n$ is a **lattice**.*

Covers of the right weak order :

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$$31245 \lessdot 31425$$

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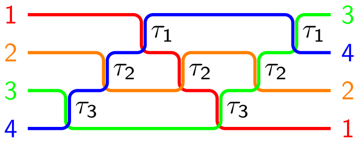
Crossing network $\leftrightarrow$ simple reflections product



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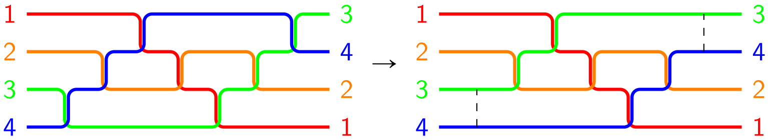
Reduction to minimal length :



Fix Q word on S, $\omega \in \mathfrak{S}_n$
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- base set : indices of Q
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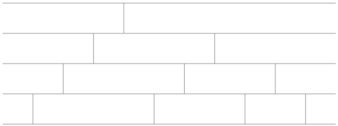

Facet $\{1, 2, 3, 8, 9\}$ of $\mathrm{SC}(\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4, 25143)$

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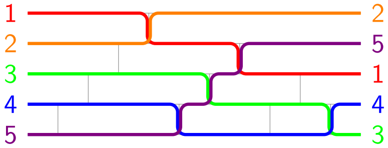

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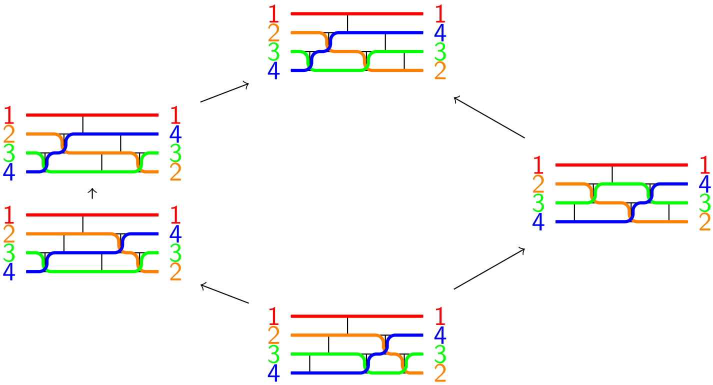
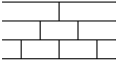
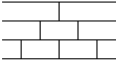
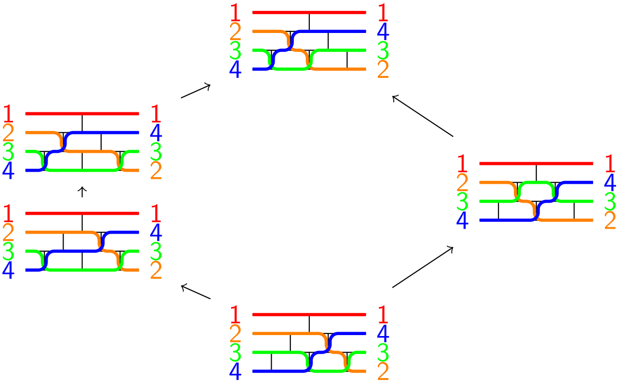
Structure given by **flips** : from one facet to another



A very special case

Q : triangular word  and $\omega = 1\, n\, (n-1)\, \ldots\, 2$

Q : triangular word  and $\omega = 1\, n\, (n-1)\, \ldots\, 2$



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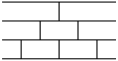
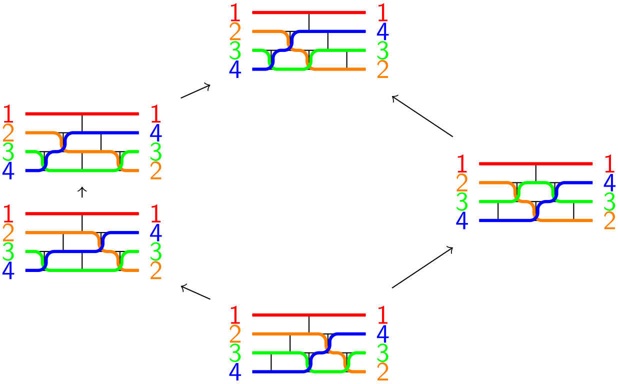
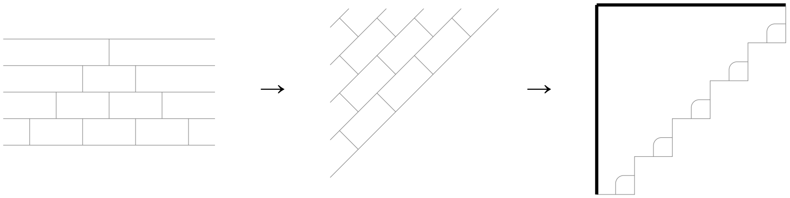
Q : triangular word  and $\omega = 1\ n\ (n-1)\ \dots\ 2$



$\Rightarrow$ this is the Tamari lattice !

Why the Tamari lattice ?



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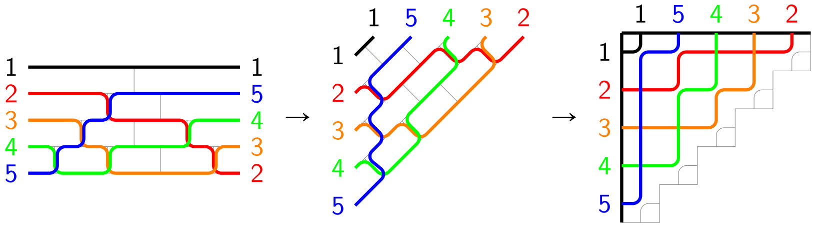

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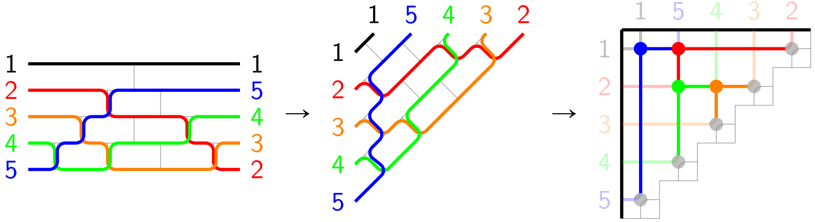

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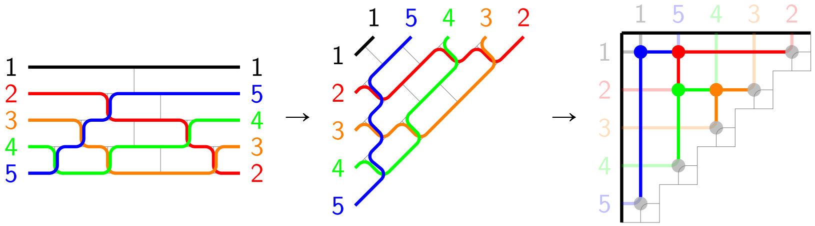

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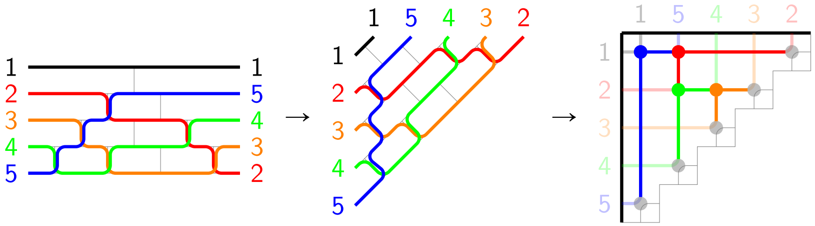

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Tree rotations ≡ flips → lattice isomorphism (Woo, 2004)

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Can we find other lattice quotients of parts of the weak order with pipe dreams ?

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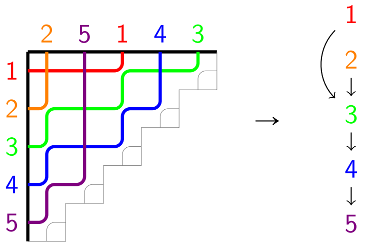

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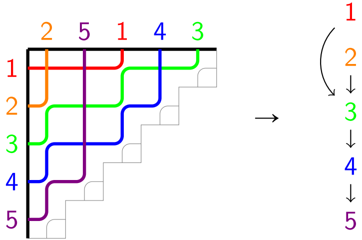

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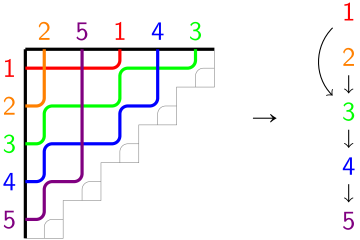

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Acyclic contact graph $\Longleftrightarrow$ vertex of the **brick polytope**

First extension : choose any permutation for the exit.

Restriction : only consider the set of acyclic pipe dreams $\Pi(\omega)$
$\rightarrow$ from permutations to pipe dreams : contact graph extensions
$\rightarrow$ domain of the application : weak order interval $[\mathrm{id}, \omega]$
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*For any $\omega \in \mathfrak{S}_n$, the set $\Pi(\omega)$ of **acyclic pipe dreams** of exit permutation ω, ordered by ascending flips, is a **lattice quotient** of the **weak order interval** $[\mathrm{id}, \omega]$.*

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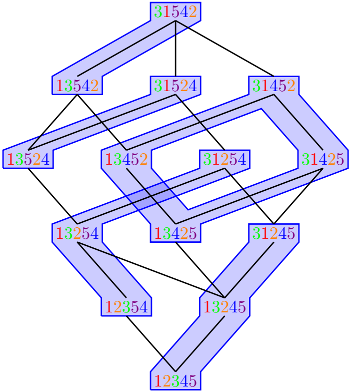
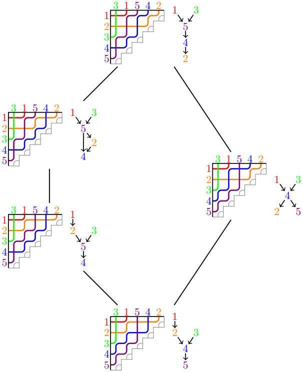
Two algorithms to compute the morphism :

- insertion algorithm (pipe by pipe)
- sweeping algorithm (cell by cell)

An example : $\omega = 31542$



$\xrightarrow{\mathsf{Ins}_\omega}$



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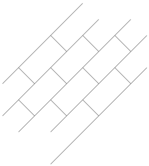

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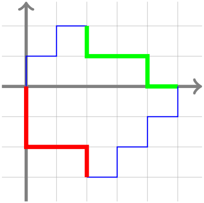

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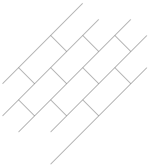

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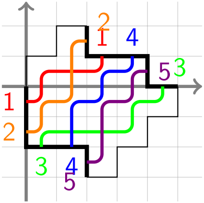

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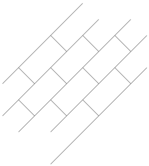

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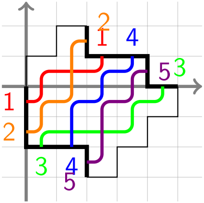

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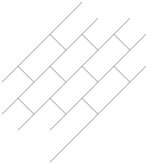

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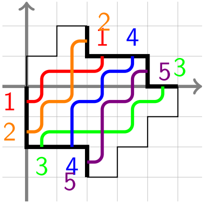

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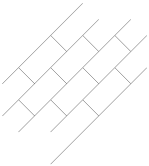

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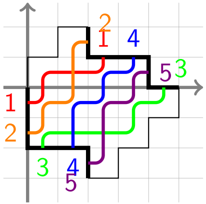

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Restrictions :

- only consider **strongly acyclic** pipe dreams
- order on pipe dreams : **acyclic order** (weaker than flip order)

Theorem (C.)

For any n-shape F and $\omega \in \mathfrak{S}_n$ sortable on F, the map $\mathrm{Ins}_{F,\omega}$ is a ***lattice morphism*** *from the* ***weak order interval*** $[\mathrm{id}, \omega]$ *to the* ***strongly acyclic*** *pipe dreams ordered by the* ***acyclic order****.*

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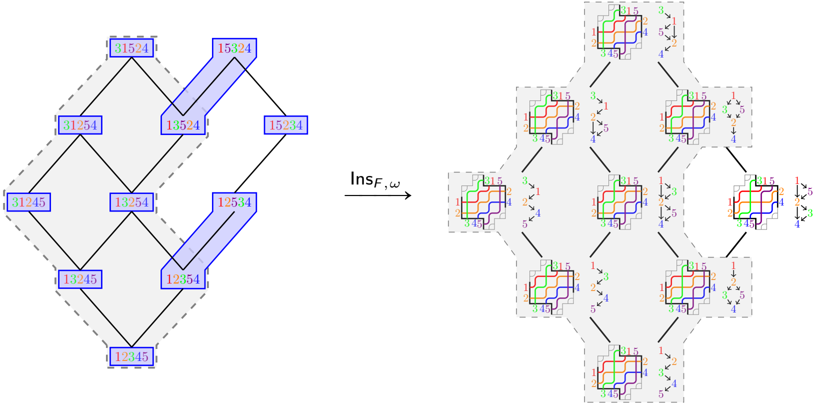
Theorem

If the maximal permutation $\omega_0 = n\,(n-1)\,\ldots 2\,1$ is sortable on F, then any linear extension of a pipe dream on F with exit permutation ω is in $[\mathrm{id}, \omega]$, and ***all acyclic pipe dreams are strongly acyclic****.*

An example : $\omega = 31524$



Further generalization : Coxeter groups

symmetric group $\mathfrak{S}_n$	Coxeter group W
simple transpositions	simple reflections
subword complexes	
pair of pipes	root in Φ
$P^{\#}$	root cone
$\pi \in \mathrm{lin}(P)$	root conf. $\subseteq \pi(\Phi^+)$

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Theorem (Jahn & Stump 2022)

If the Demazure product of Q is w_0, then for any $w \in W$ the application $\mathrm{Ins}_Q(w, \cdot)$ is ***surjective on acyclic facets*** *of $\mathrm{SC}(Q, w)$.*

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Conjecture

If Q is an alternating word on S and $w \in W$ is sortable on Q, then the application $\mathrm{Ins}_{Q,w} : [e, w] \mapsto \mathrm{SC}(Q, w)$ is a **lattice morphism** from the left weak order on $[e, w]$ to its image.

Work in progress...

Thank you for your attention !

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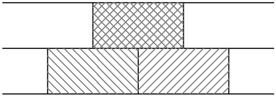
- **bricks** of Q : bounded cells



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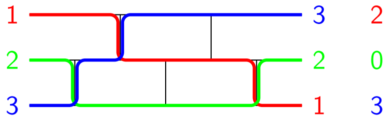

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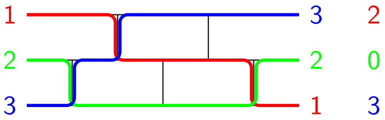

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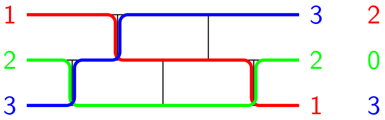

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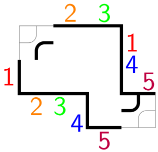
- **brick polytope** of $\mathrm{SC}(Q, \omega)$: convex hull of brick vectors of facets

Sweeping algorithm for $\omega = 23145$ and $\pi = 21345$



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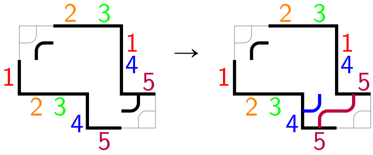

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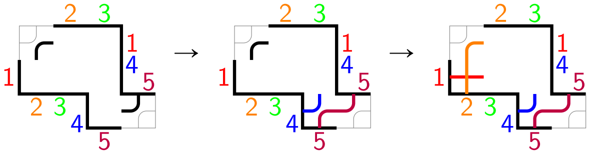
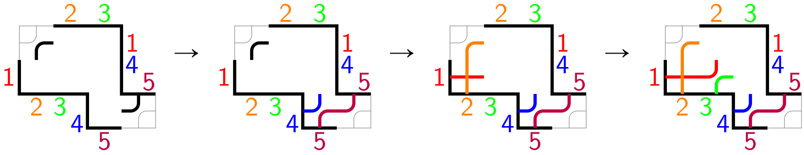


Sweeping algorithm for $\omega = 23145$ and $\pi = 21345$



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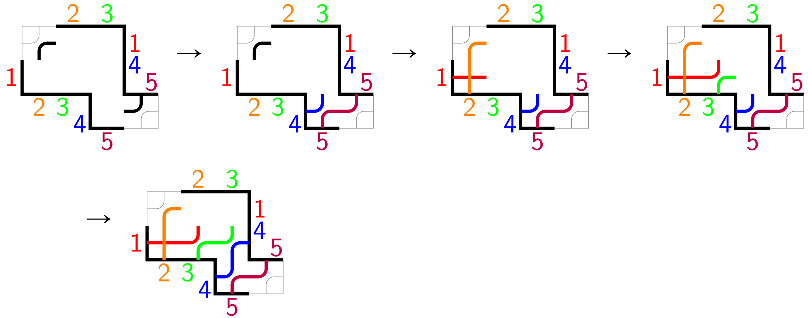

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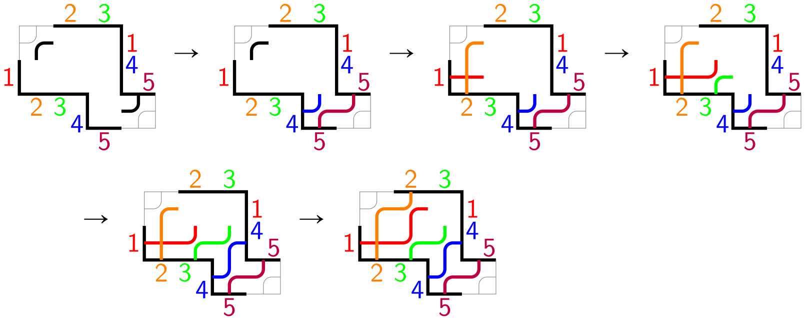

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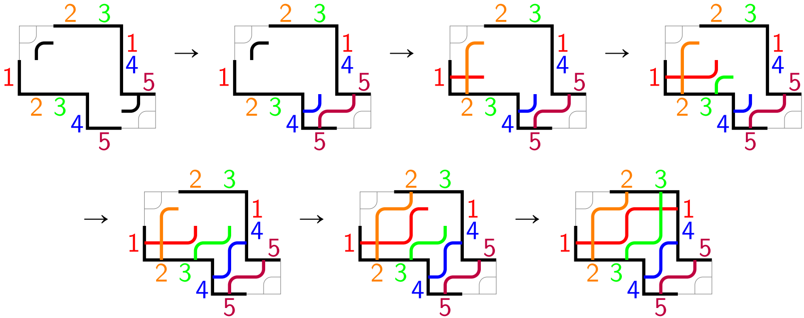

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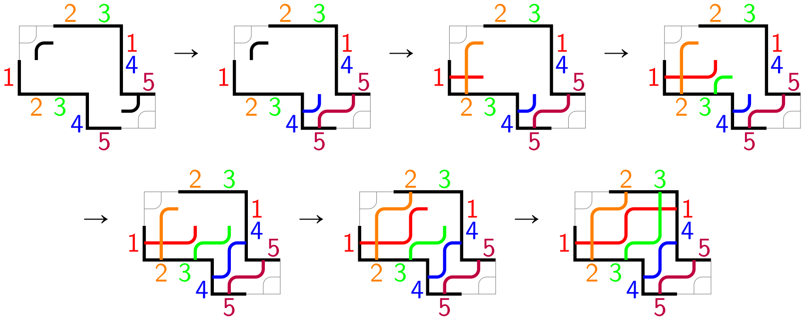

1. if $\omega^{-1}(i) < \omega^{-1}(j)$, add an elbow
2. if $\omega^{-1}(i) > \omega^{-1}(j)$ and $\pi^{-1}(i) > \pi^{-1}(j)$, add a cross +
3. if i, j inversion of ω and non-inversion of π, add an elbow if you can still make the pipes end in order ω that way (3a), and a cross + otherwise (3b)

Insertion algorithm for $\omega = 3241$ and $\pi = 2134$
The idea : keep track of cells containing only half of an elbow, and complete as many of those cells as possible when inserting a new pipe.

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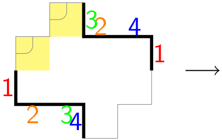

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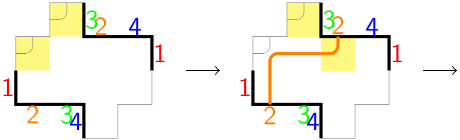

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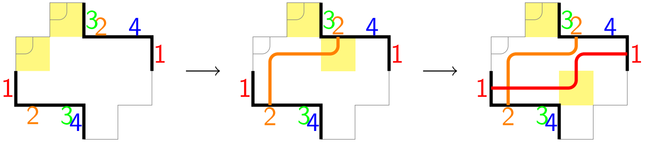

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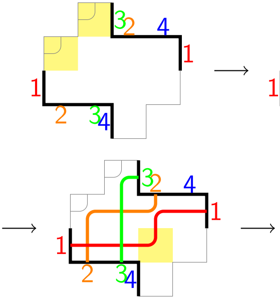

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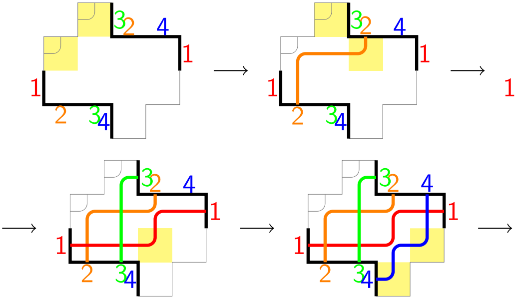

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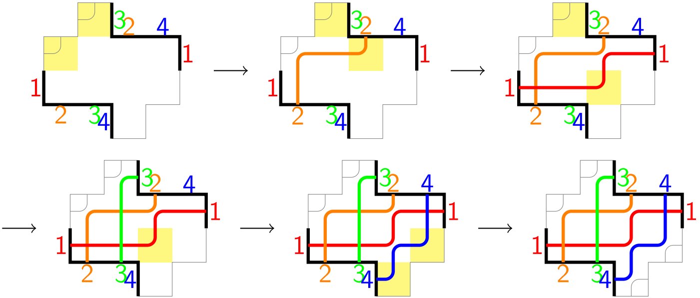
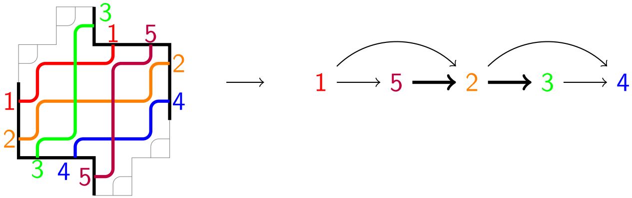


An acyclic but not strongly acyclic facet :



One linear extension : $15234 \not\leq 31524$.