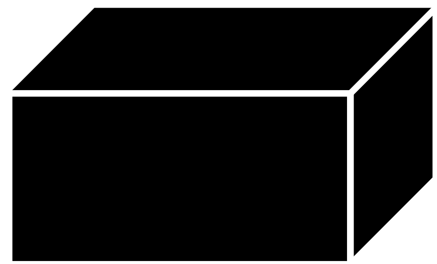
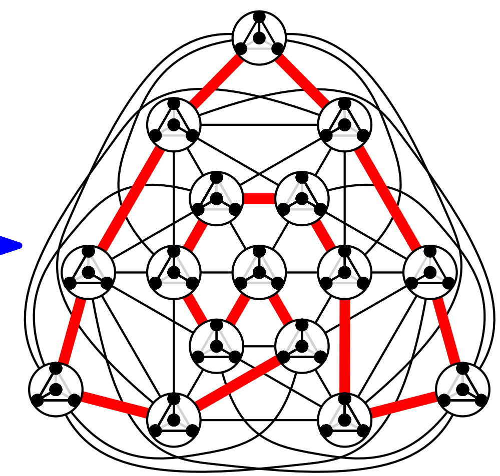
Traversing Combinatorial 0/1-Polytopes

$$
\begin{aligned}
\min \quad & d_H(x^*, x) \\
\text{s.t.} \quad & x \in \mathcal{X}, \\
& x_i = 1 \quad \forall i \in I_P, \\
& x_i = 0 \quad \forall i \in I_F.
\end{aligned}
$$

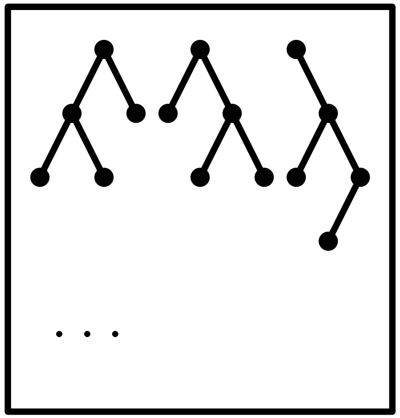




Arturo Merino
Saarland University & Max Planck Institute for Informatics
Joint work with Torsten Mütze

Introduction

- Many different classes of combinatorial objects



binary trees

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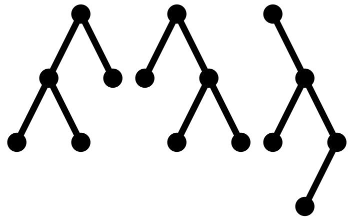

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binary trees

123
132
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permutations

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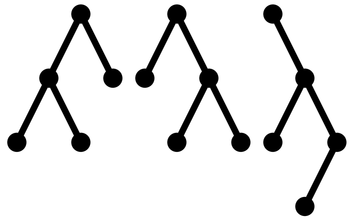

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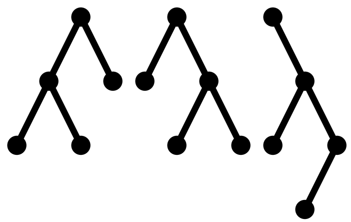

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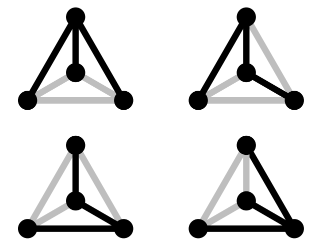

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spanning trees

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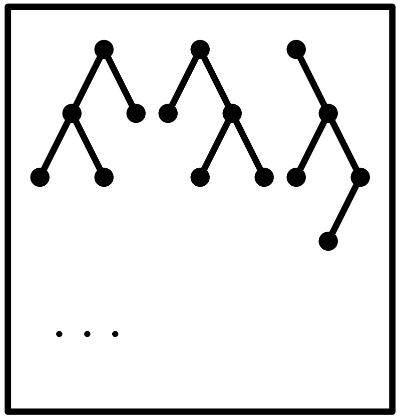

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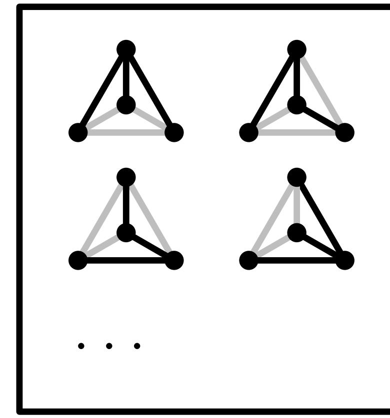

spanning trees

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Introduction

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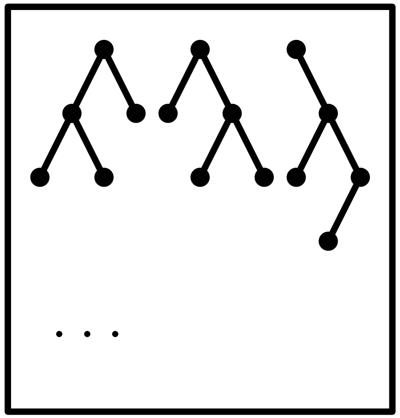

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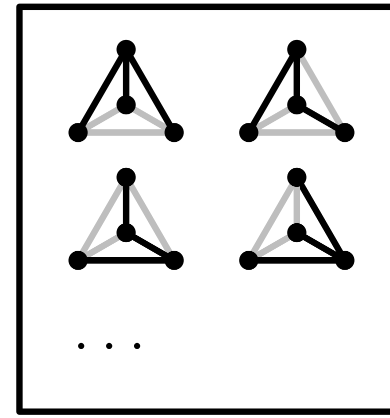

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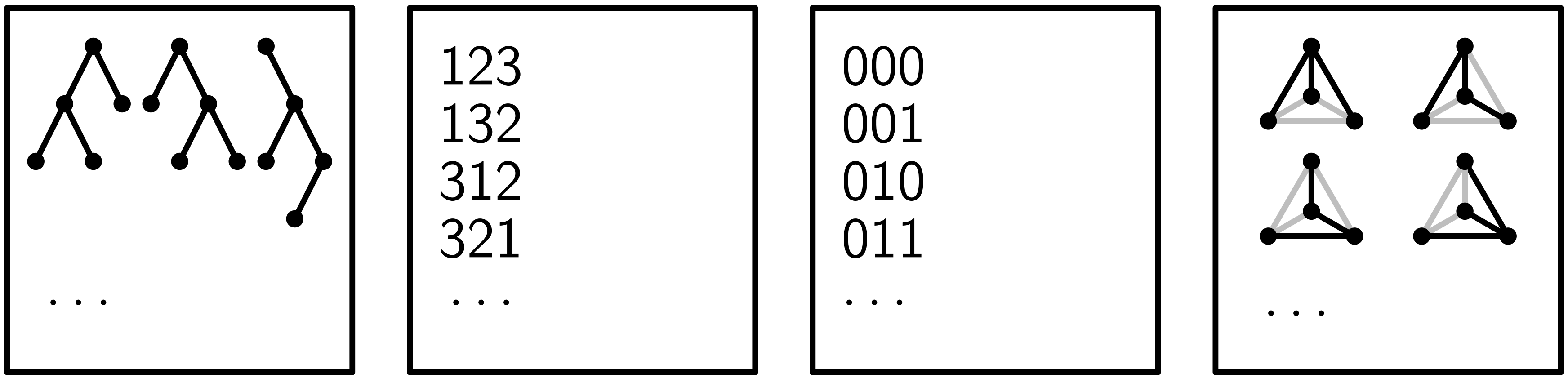

binary trees | permutations | bitstrings | spanning trees

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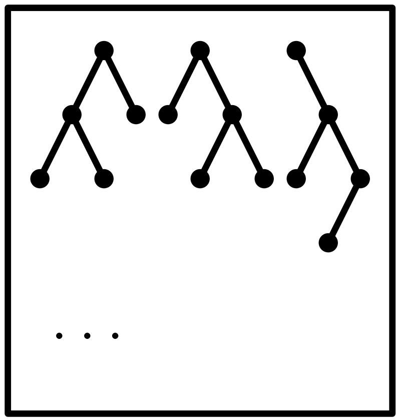

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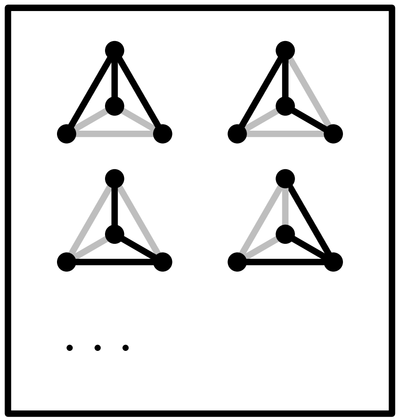

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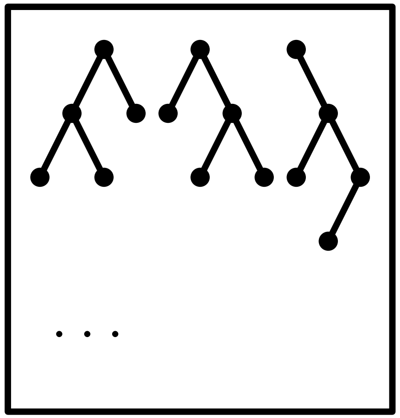

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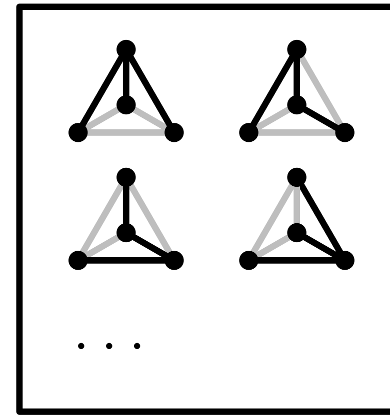

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- fundamental algorithmic tasks:
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    - **combinatorial generation** **[Knuth TAOCP Vol. 4A]**.

Combinatorial generation

- **Goal:** generate all objects of a combinatorial class efficiently.

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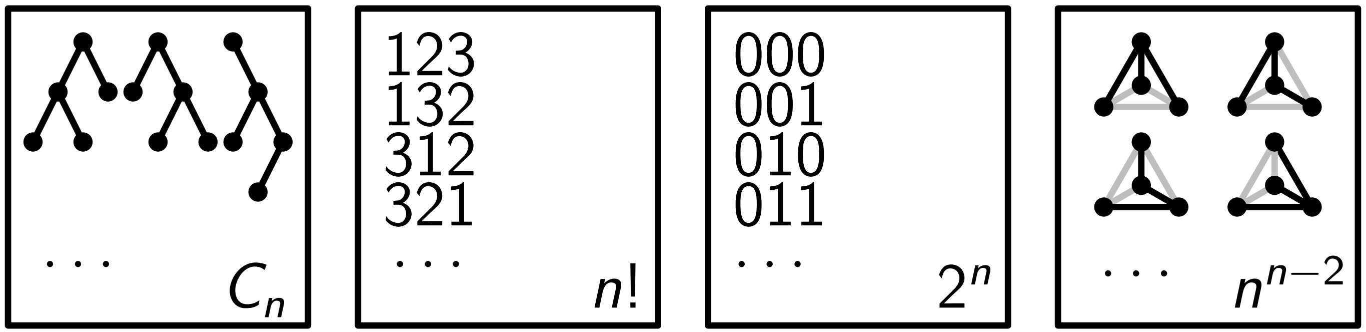

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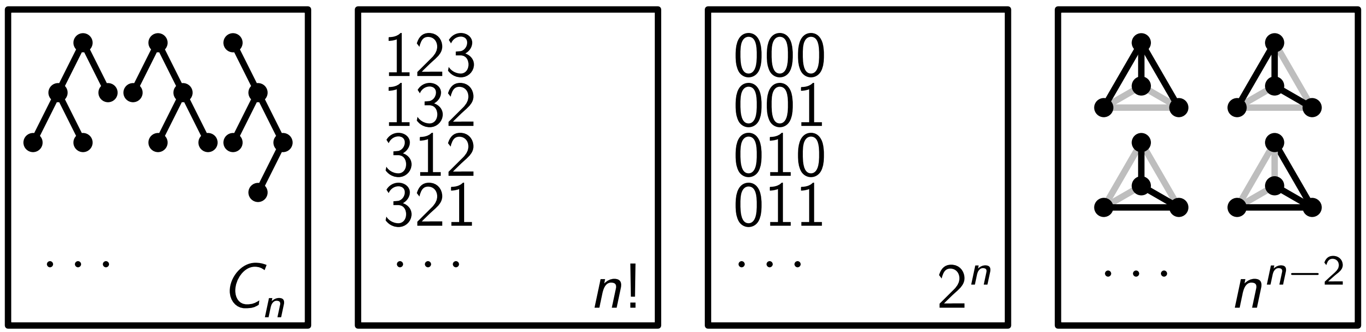

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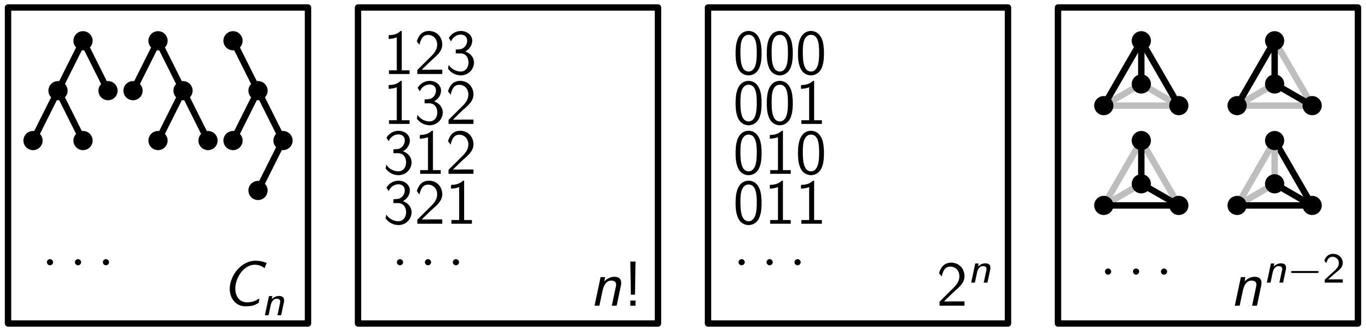

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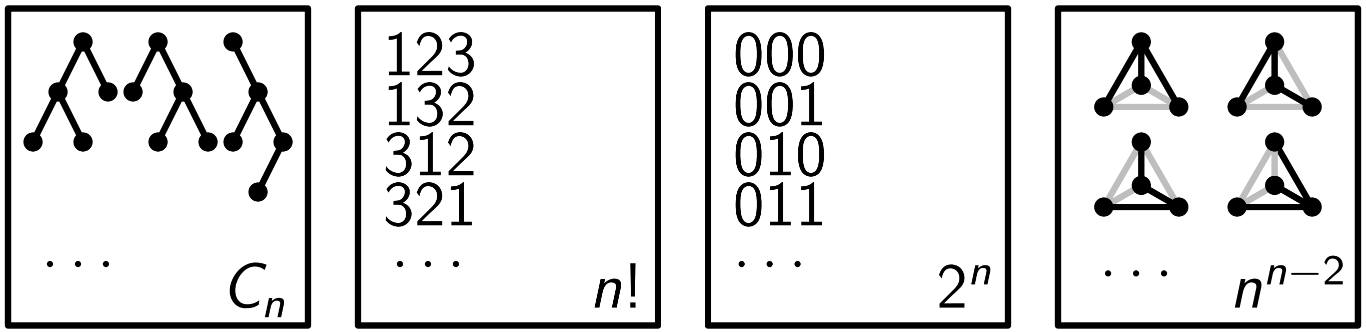

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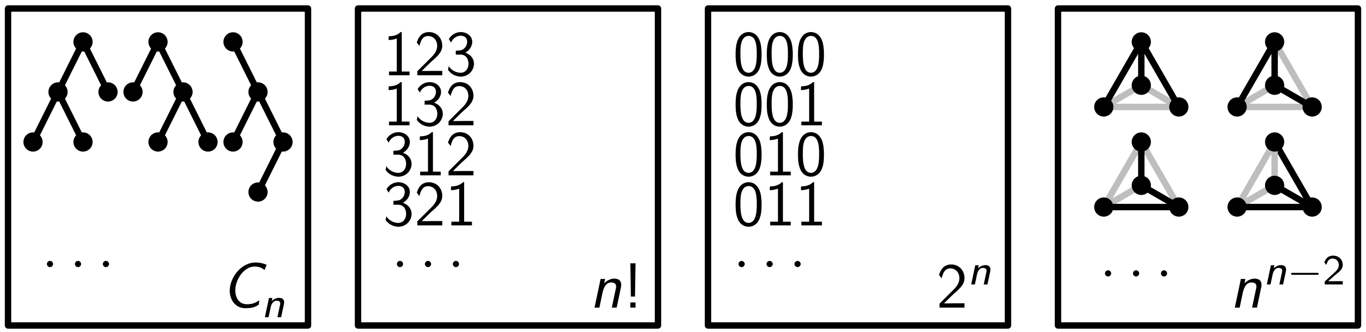

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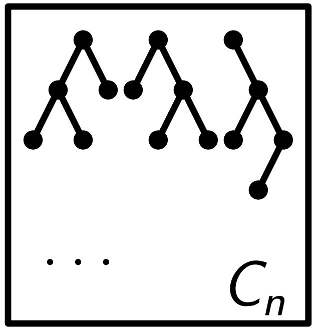

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$n!$

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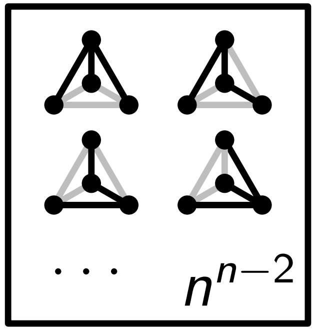
2^n



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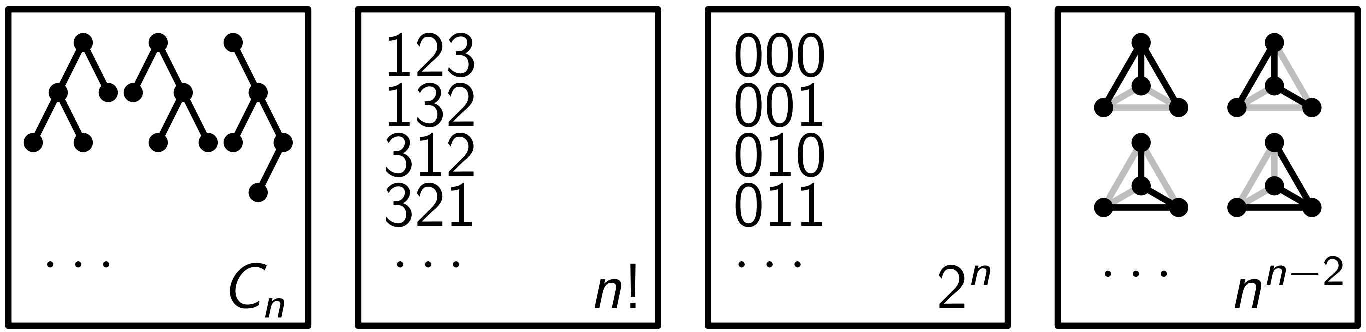

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- Listing where consecutive objects differ by a **"local change"**
  $\rightarrow$ **Gray code** (recent survey **[Mütze 22]**)

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- All these examples lead to constant (amortized) delay generation algorithms.

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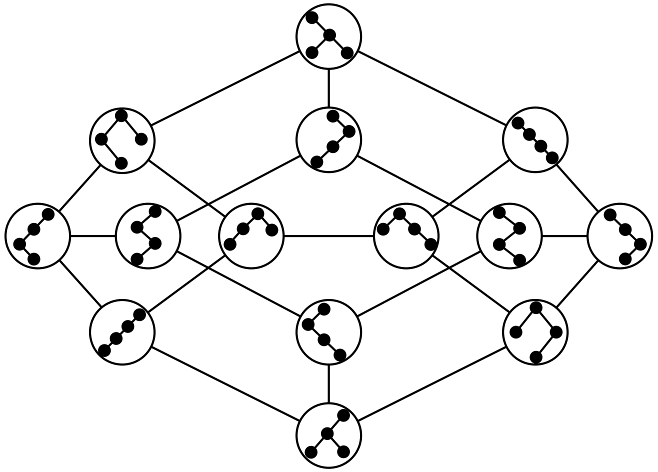

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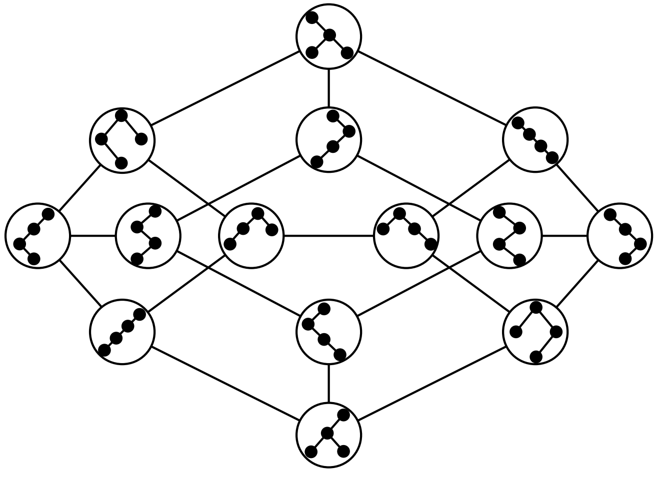

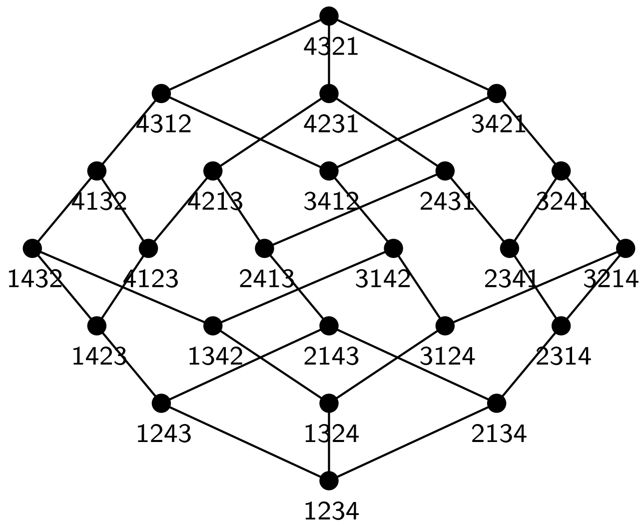

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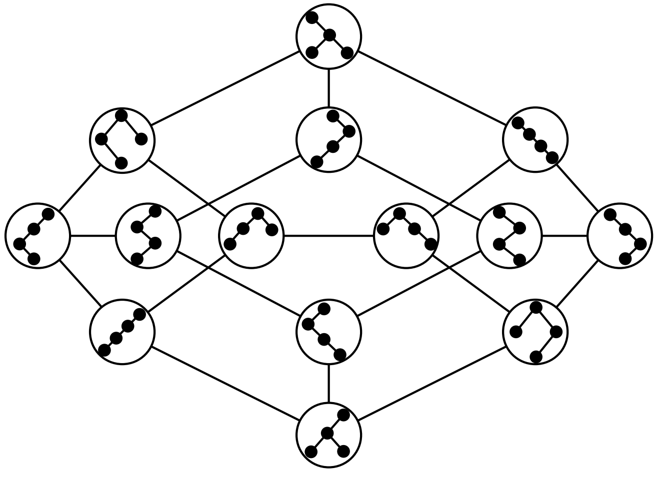

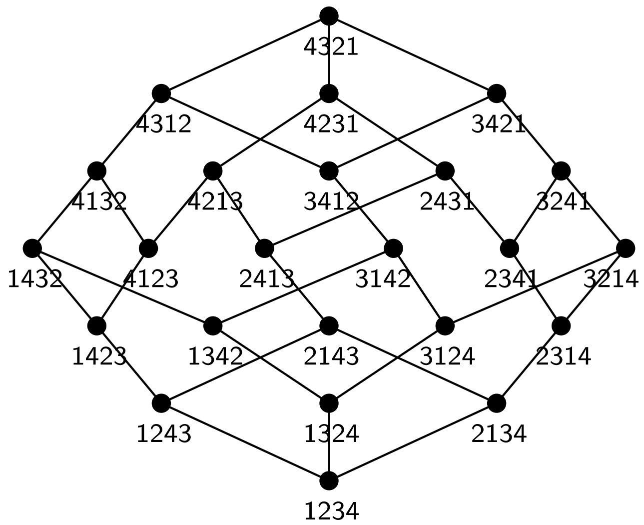

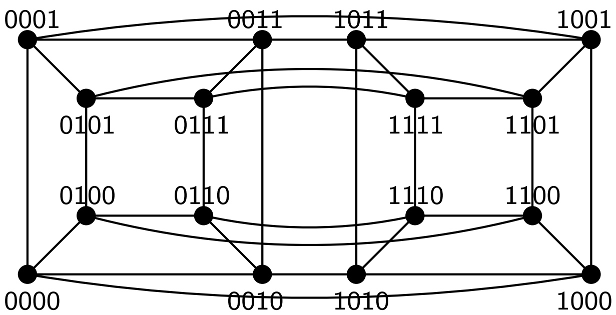

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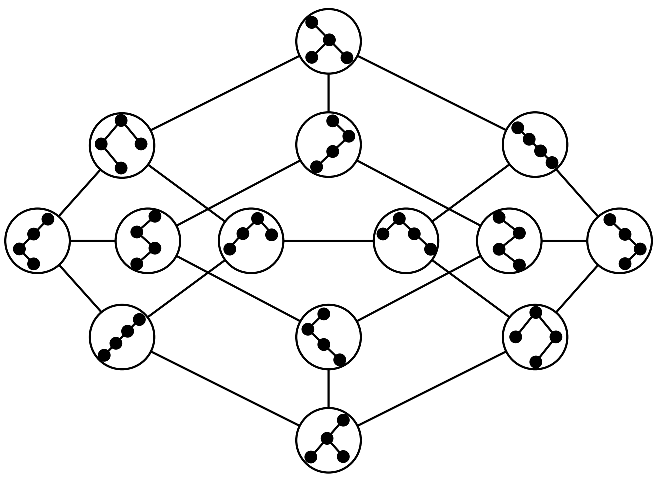
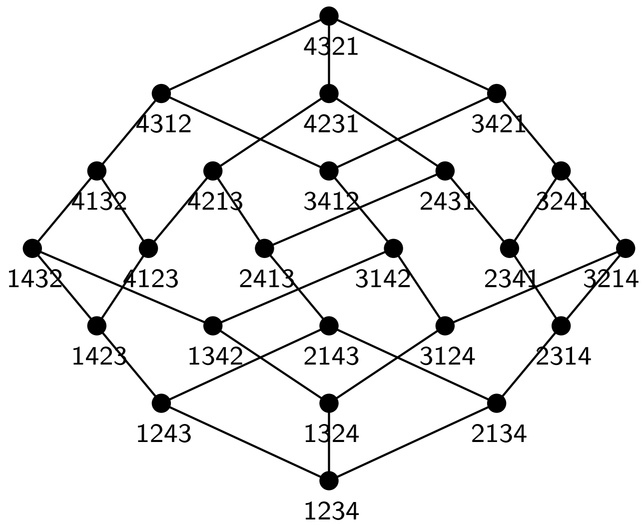
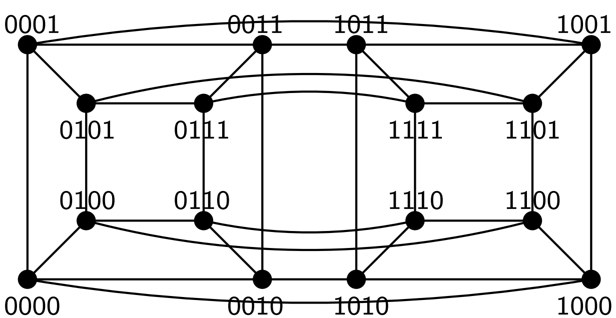
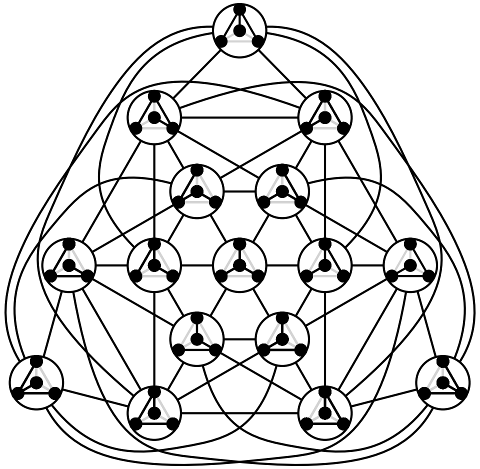


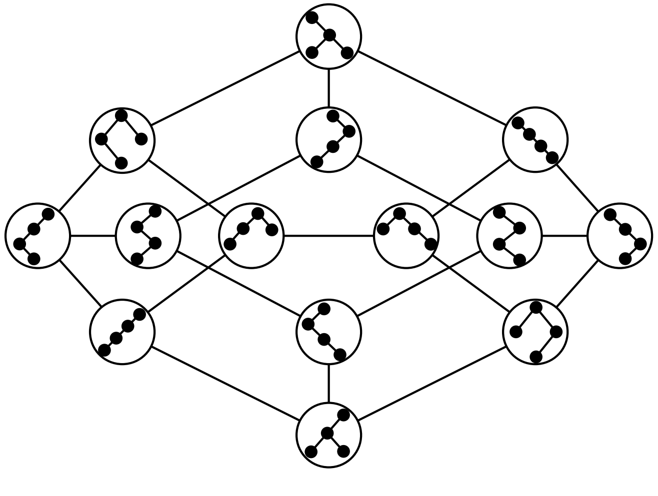


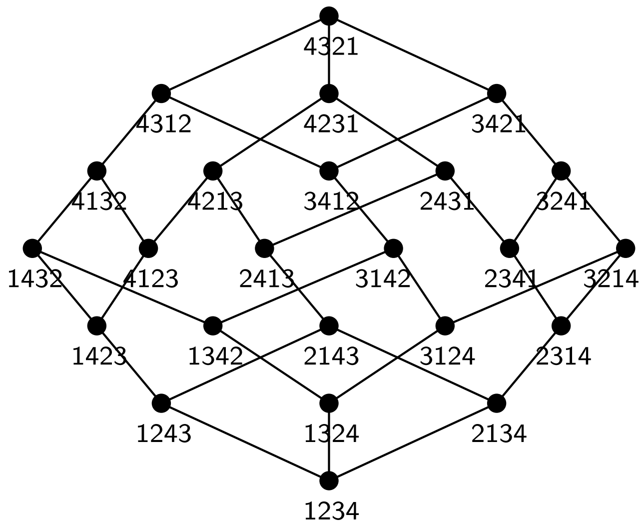


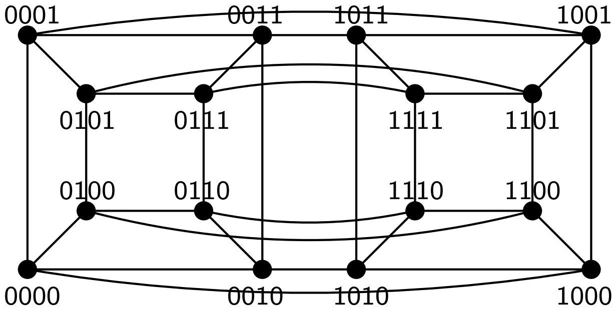


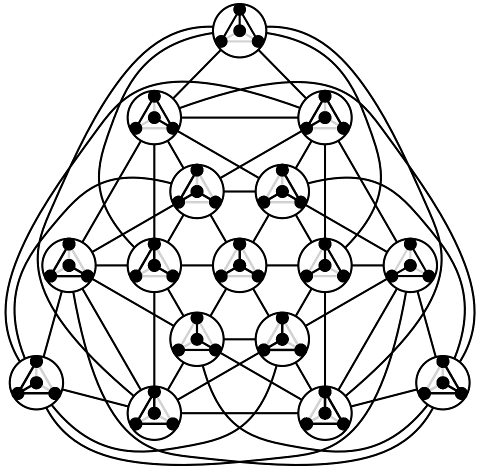
Gray Codes and Flip Graphs

- Many flip graphs can be realized as polytopes and posets.









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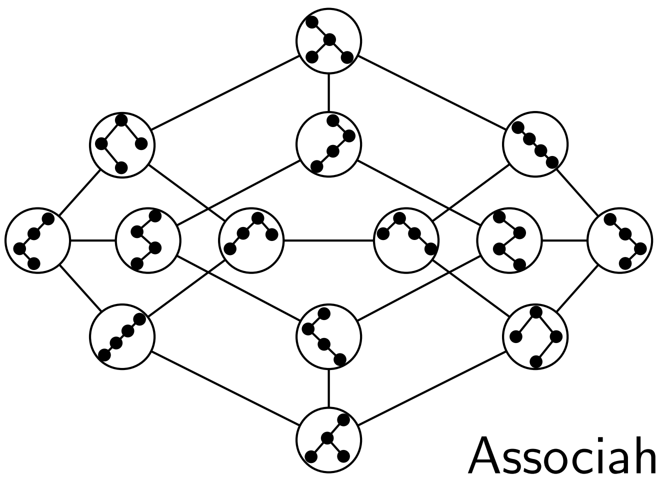
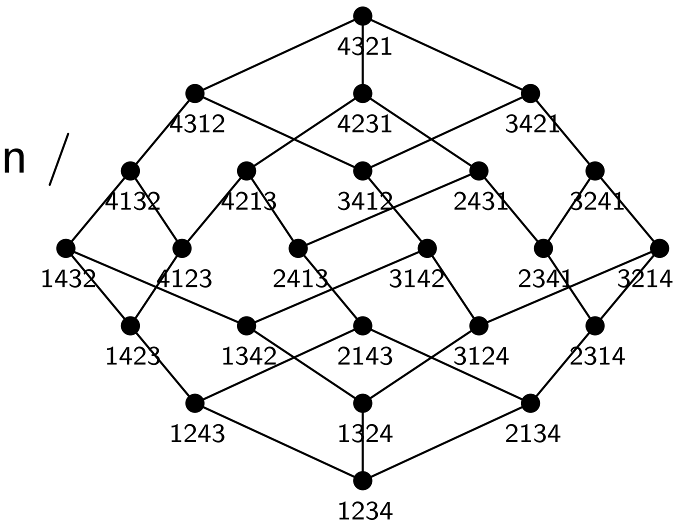
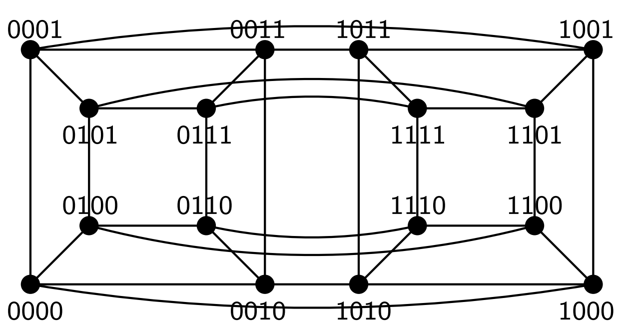


Associahedron / Tamari lattice

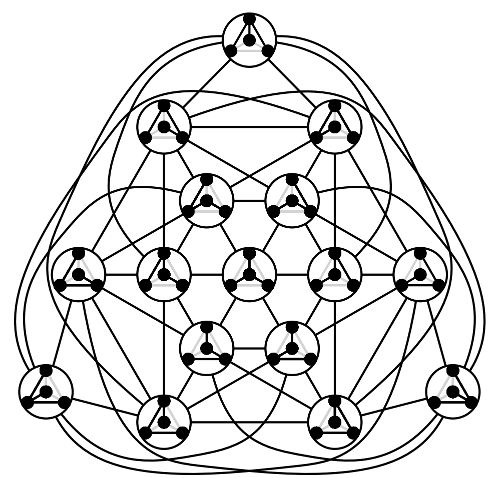
Π_4 Permutahedron / weak order





Q_4 Hypercube / Boolean lattice

Base polytope



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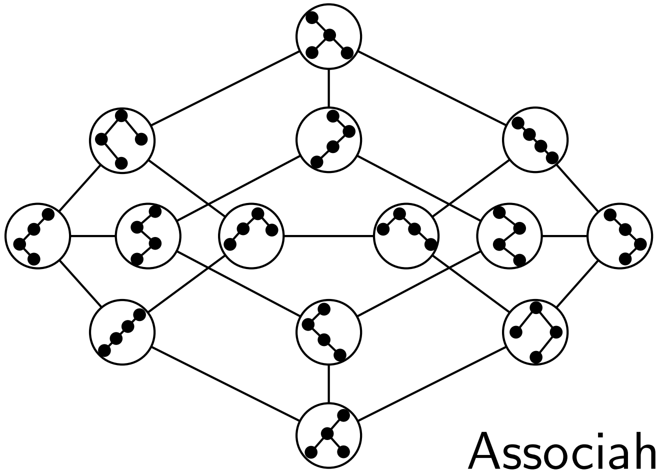
- Gray code → **Hamilton path/cycle** on the flip graph



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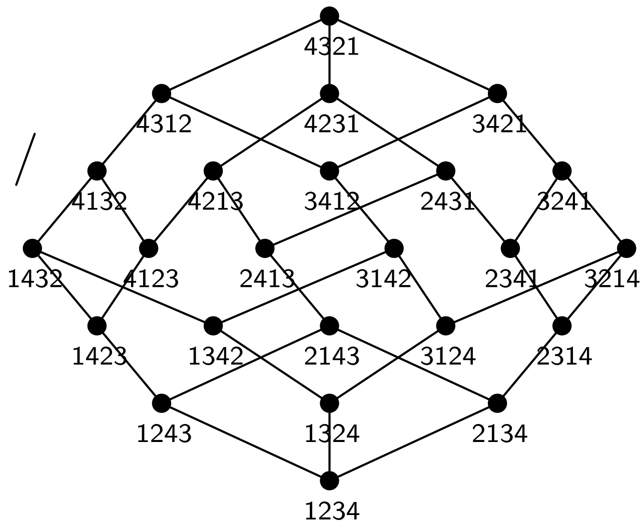

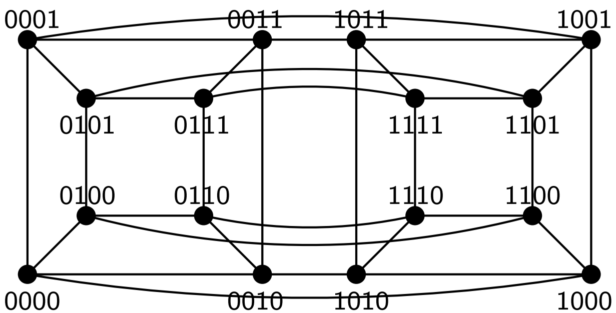

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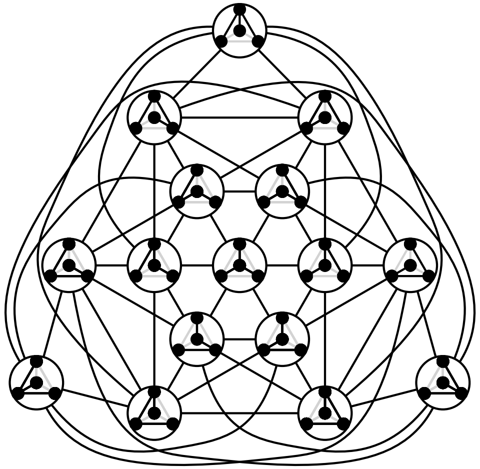

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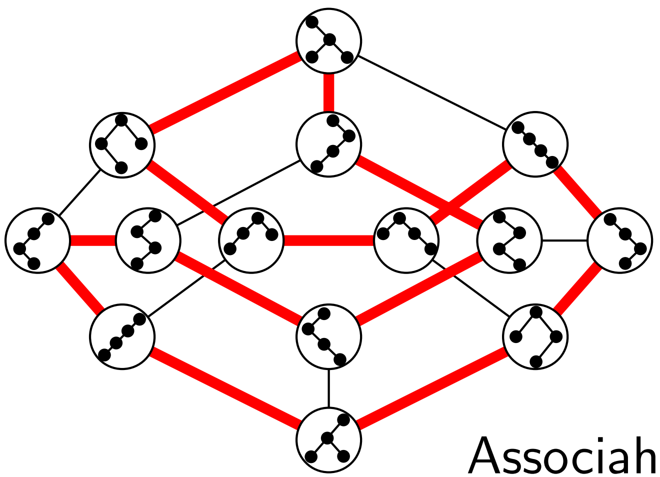

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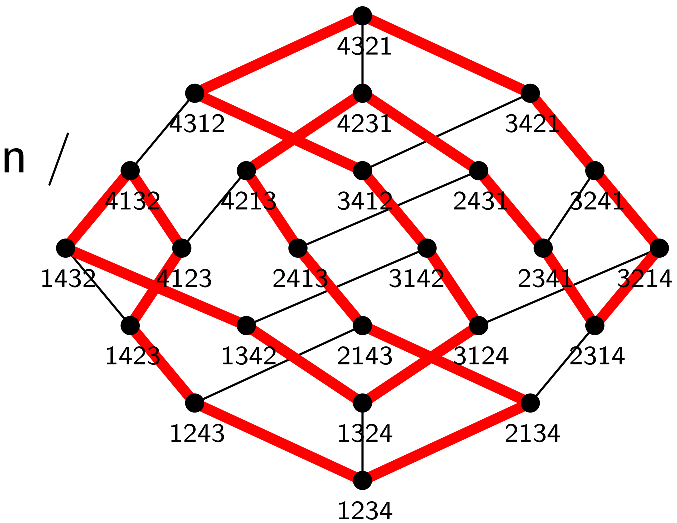

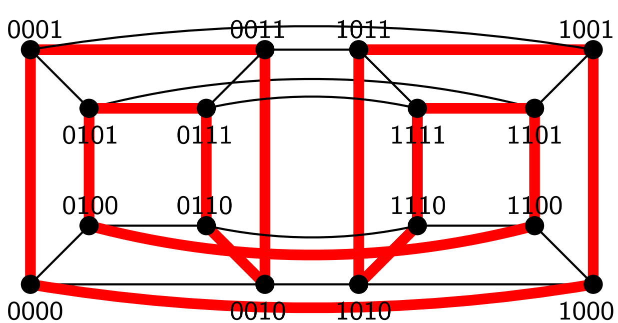

Q_4 Hypercube / Boolean lattice

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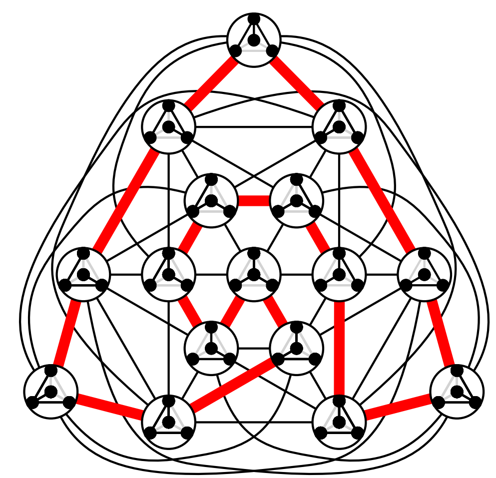

Binary sets and 0/1 polytopes

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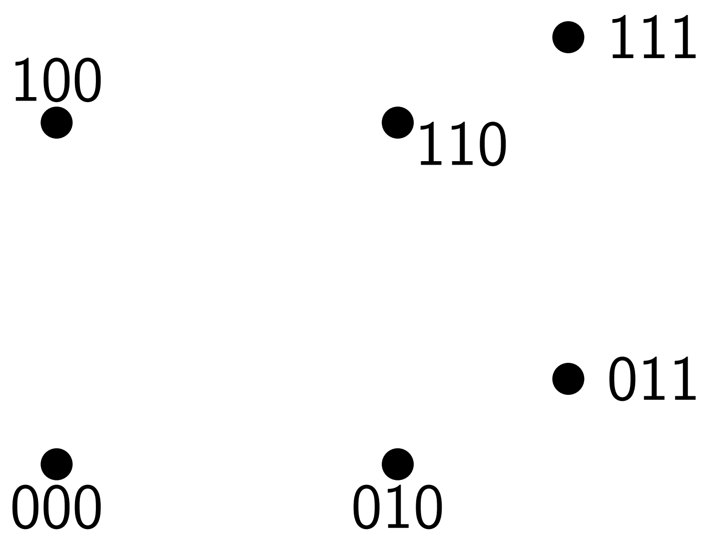

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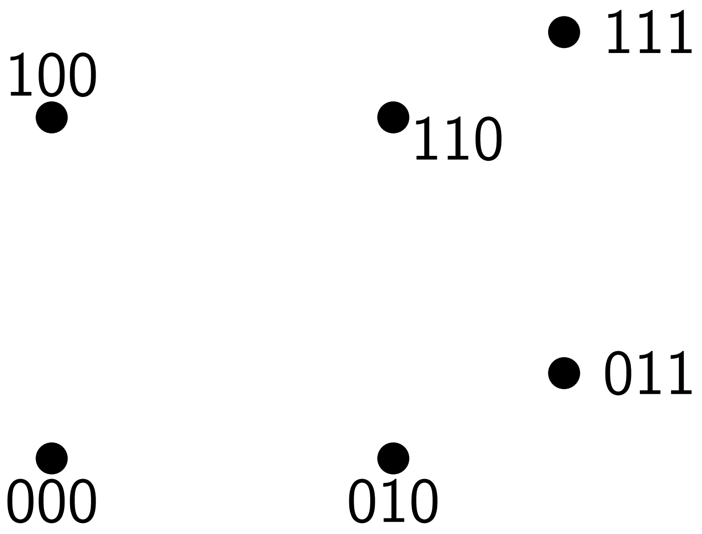

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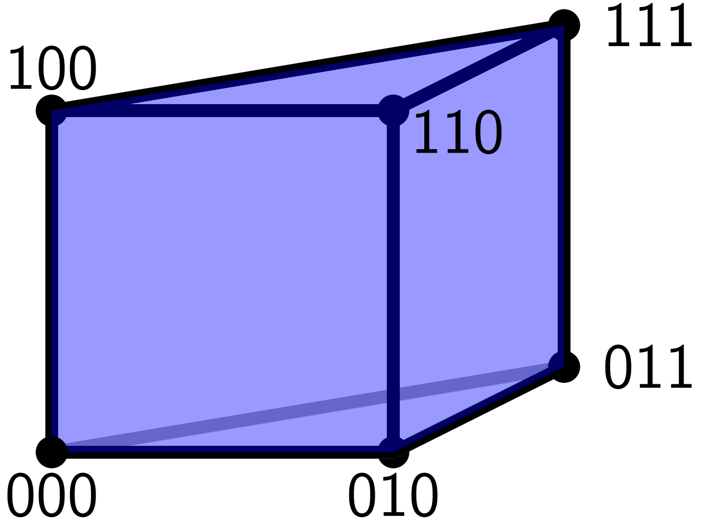

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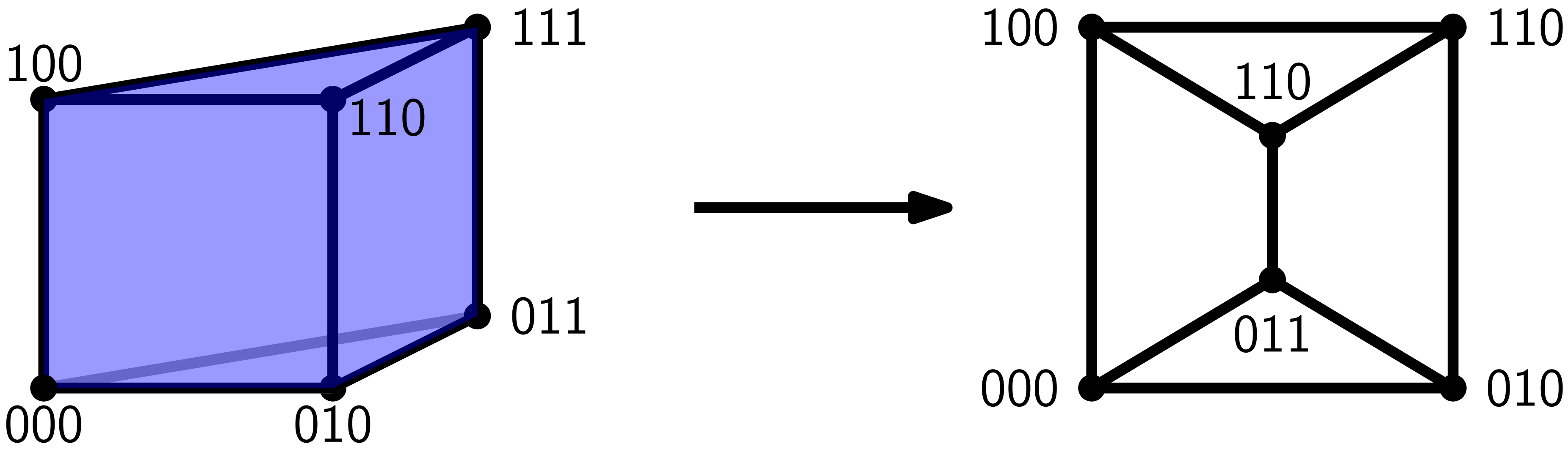

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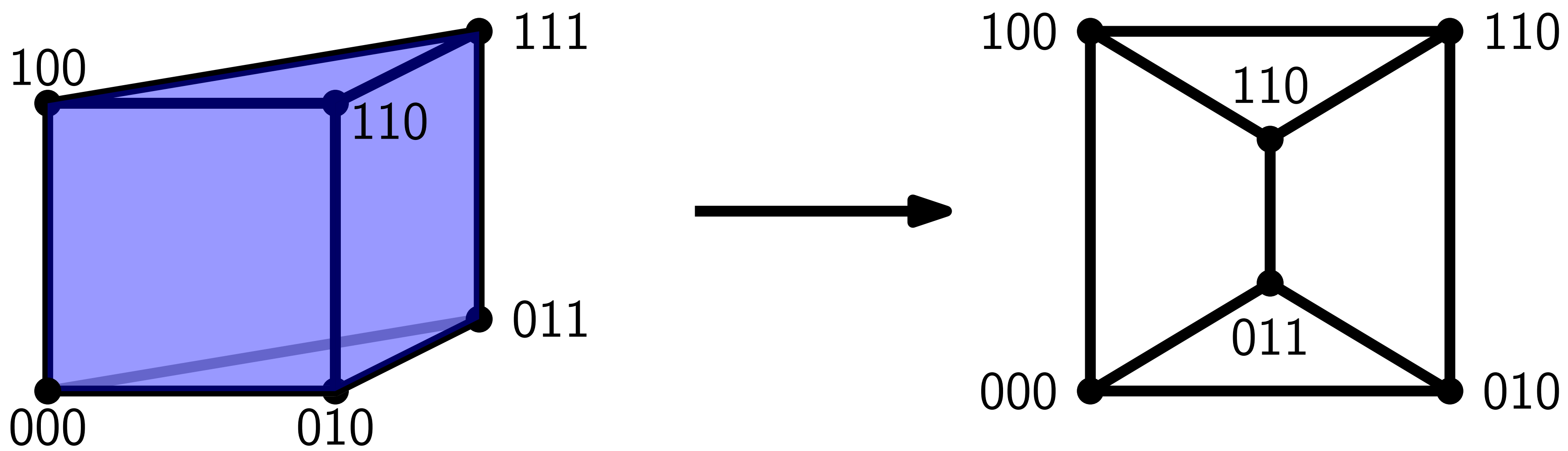

- The edges of conv($\mathcal{X}$) capture "local changes" betwen elements of $\mathcal{X}$.
- Hamilton paths in conv($\mathcal{X}$) are combinatorial Gray codes.

Four 0/1 problems: Vertex enumeration for 0/1 polytopes

0/1 Vertex enumeration problem

Given: Inequalities $Ax \leq b$ that define a 0/1 polytope.

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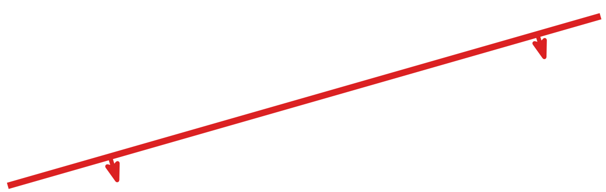

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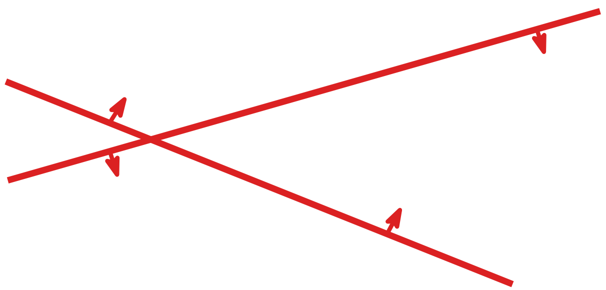

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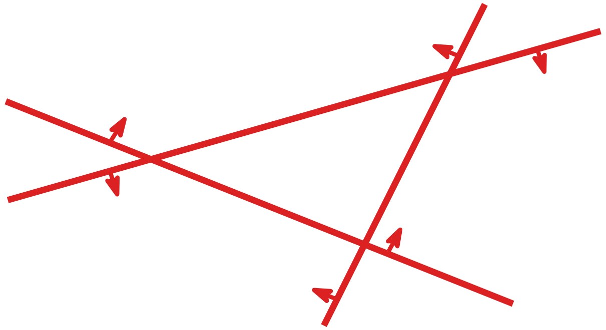

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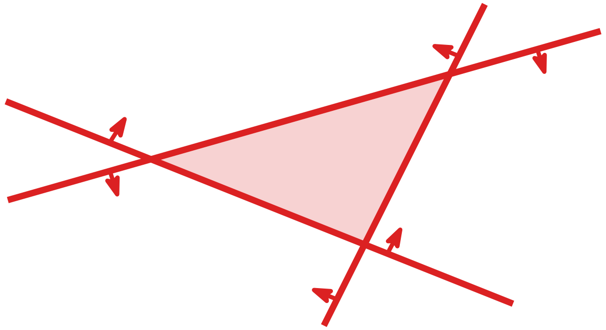

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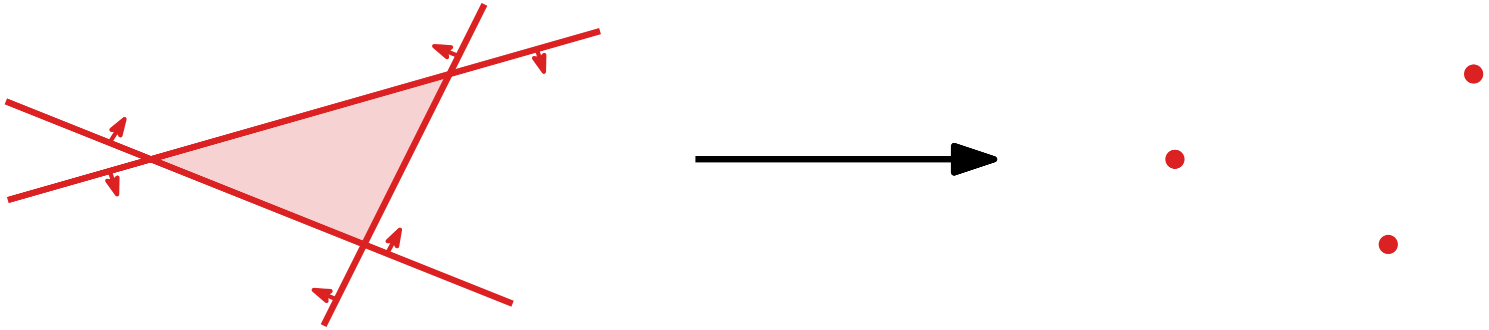

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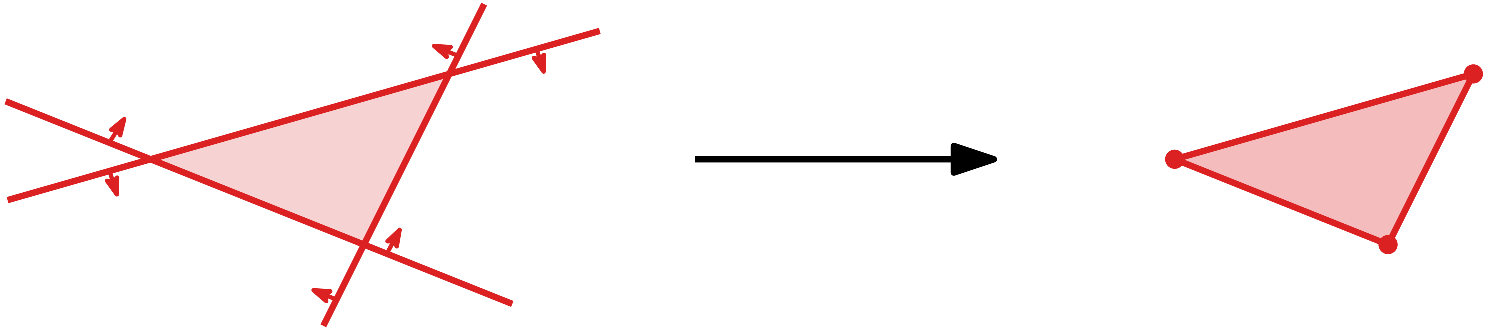

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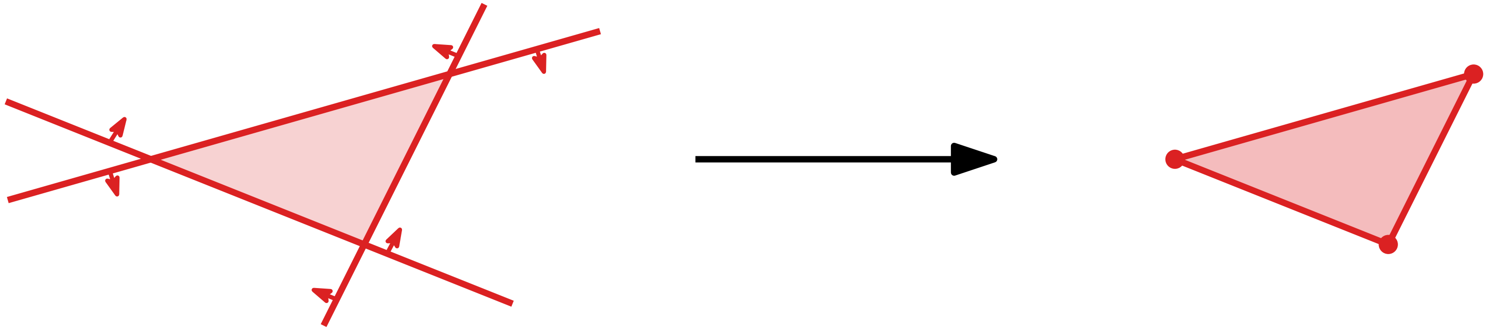

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Avg. delay Delay		

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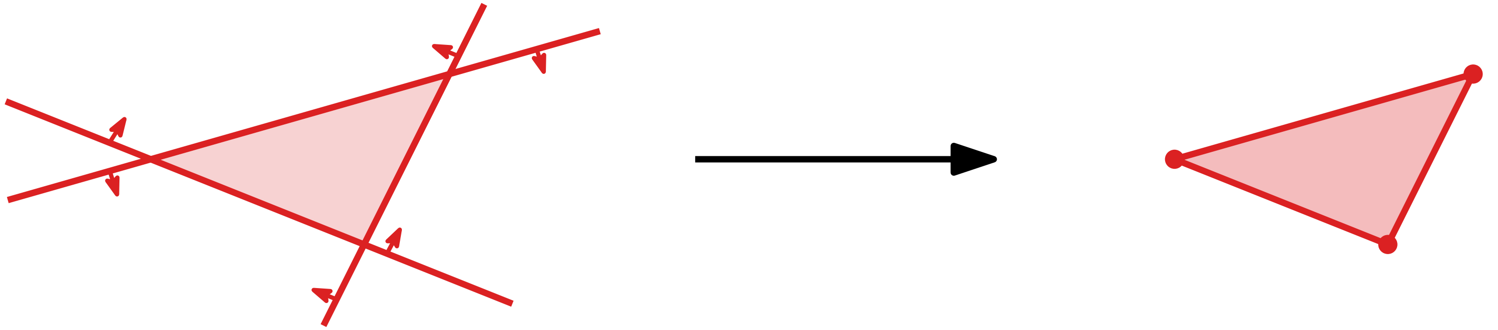

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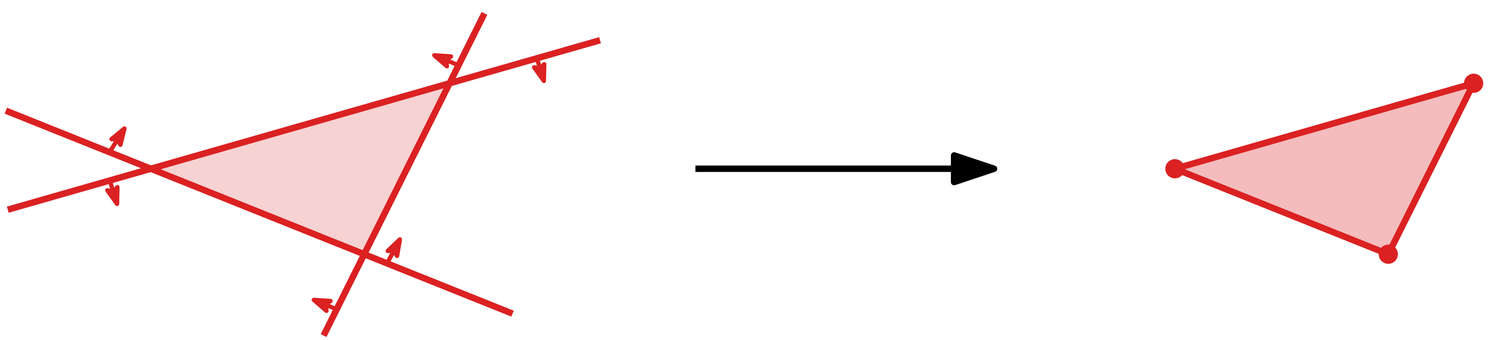

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Main results

Thm.

Optimization over $\mathcal{X} \subseteq \{0,1\}^n$ can be solved in time $\mathcal{O}(T)$

$$\Longrightarrow$$

A Hamilton path in $\mathrm{conv}(\mathcal{X})$ can be computed in $\mathcal{O}(T \operatorname{polylog} n)$-delay.

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Black box corollaries

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Results in non-bipartite graphs!

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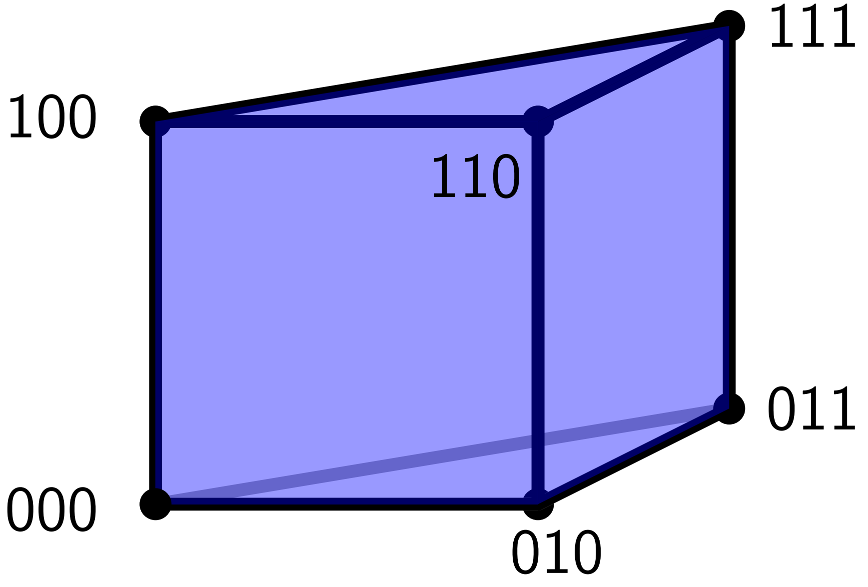

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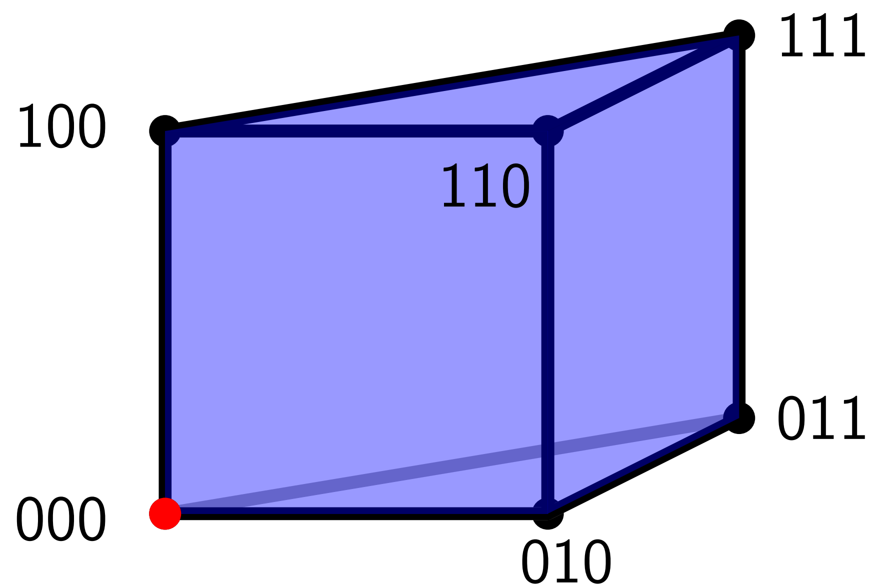

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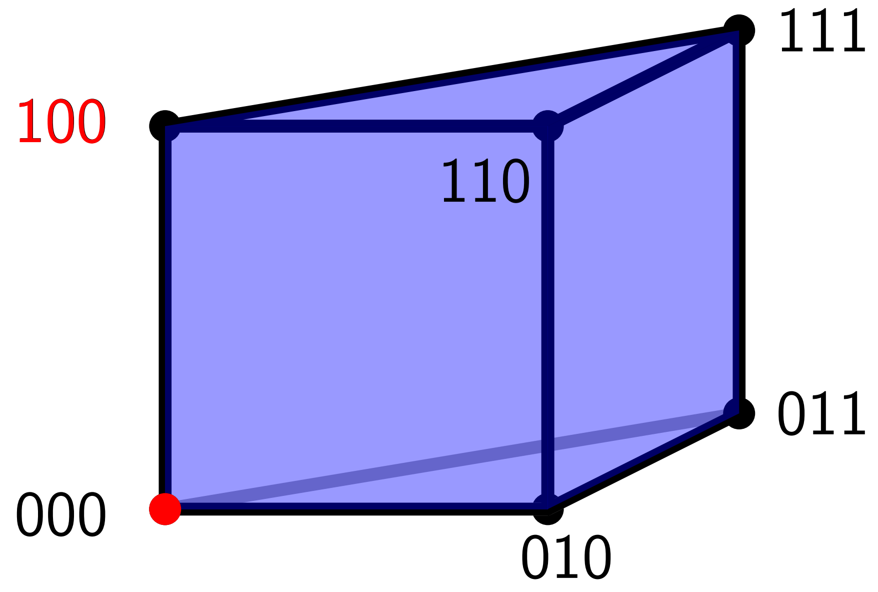

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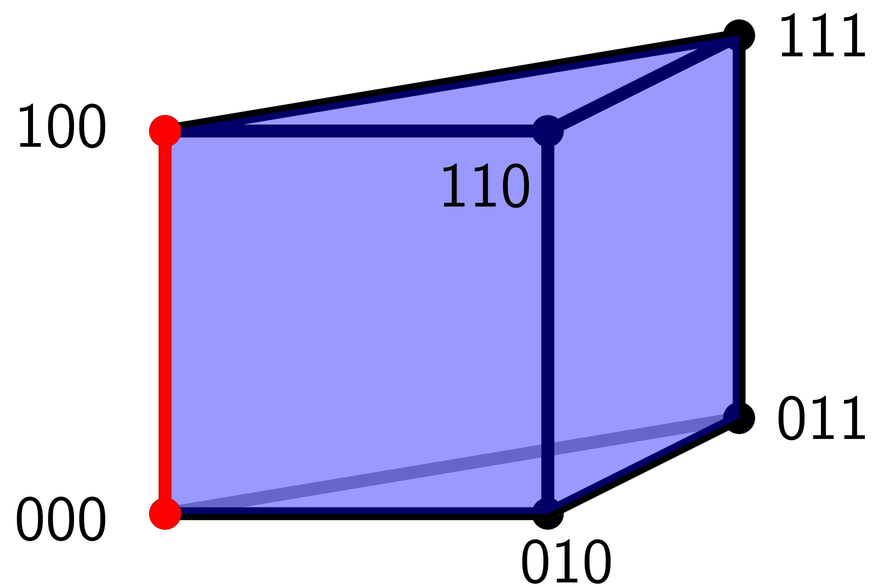

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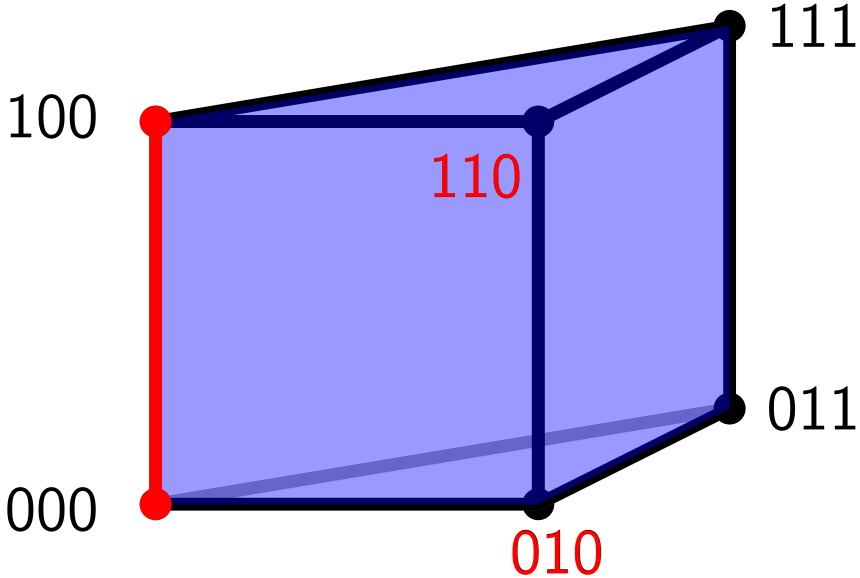

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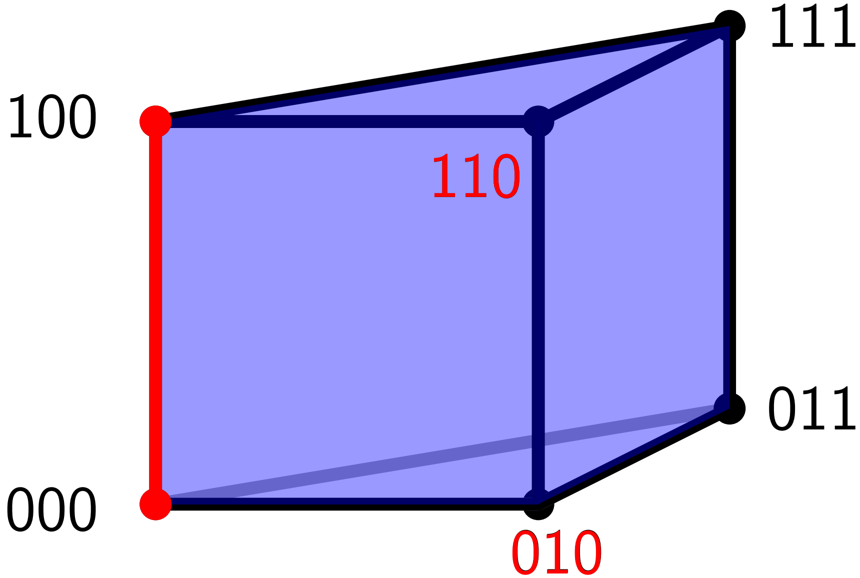

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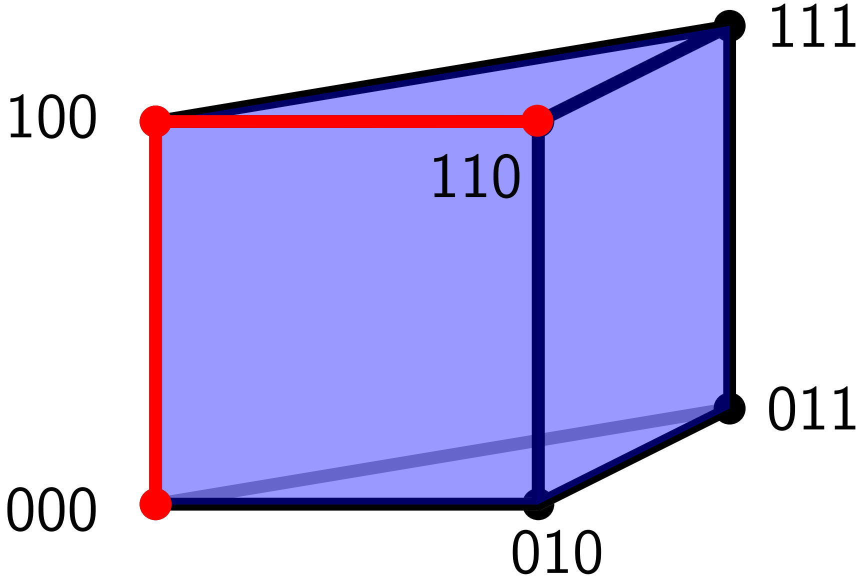

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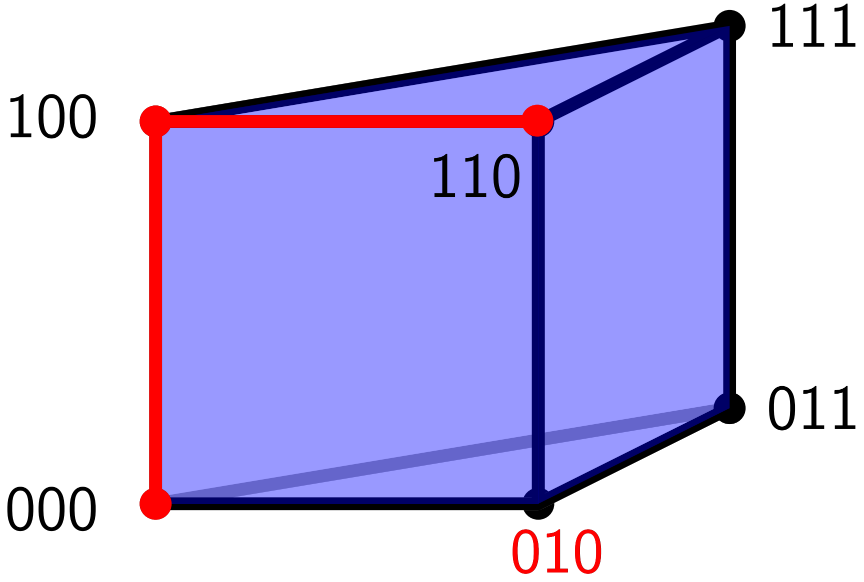

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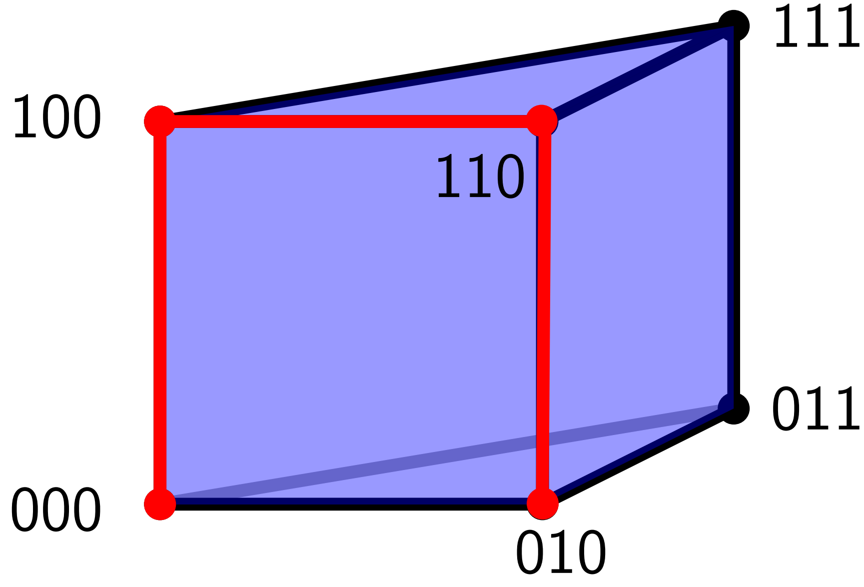

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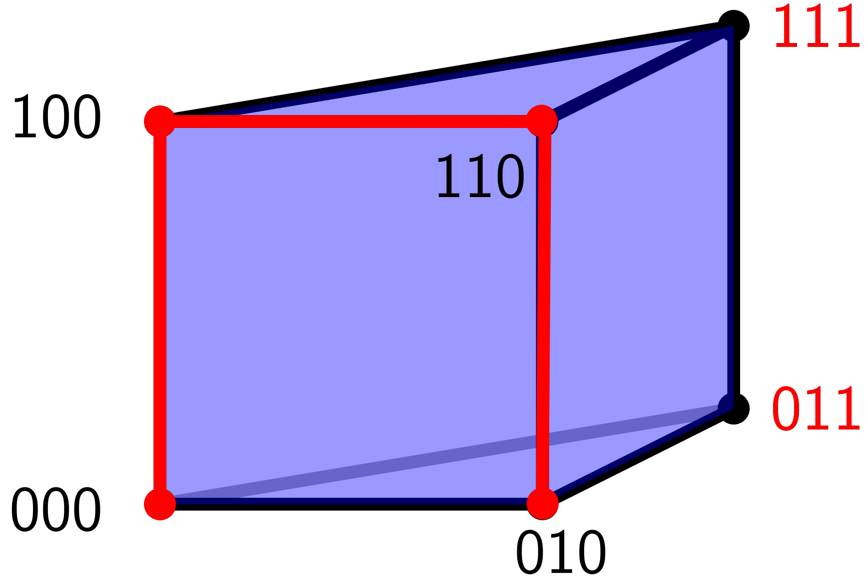

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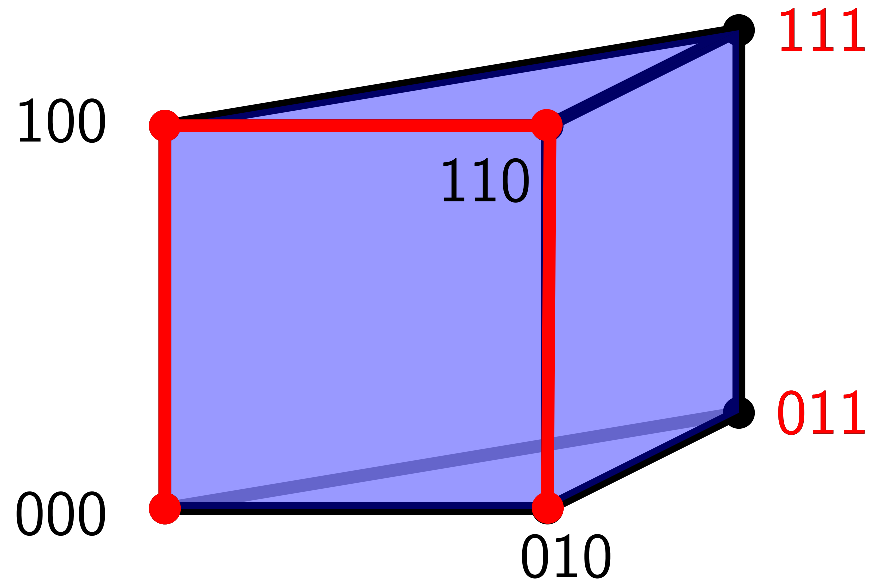

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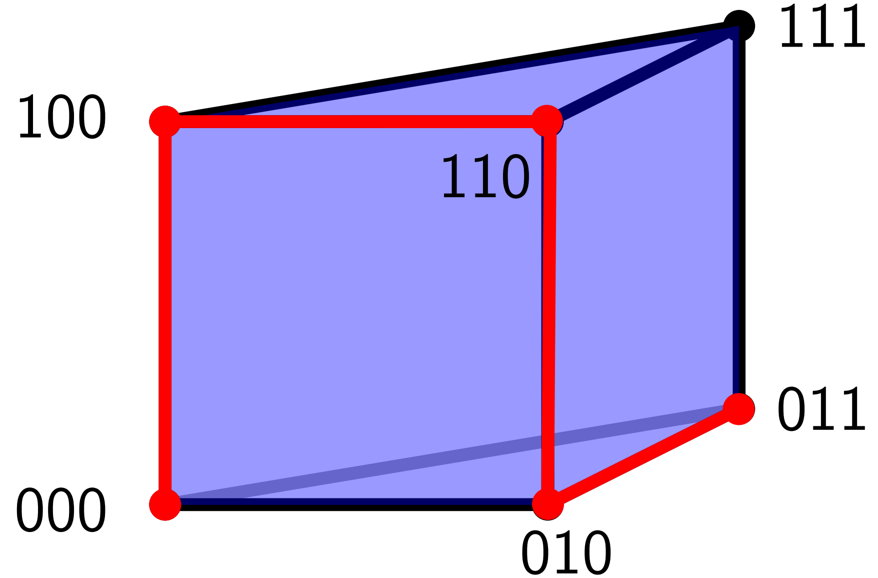

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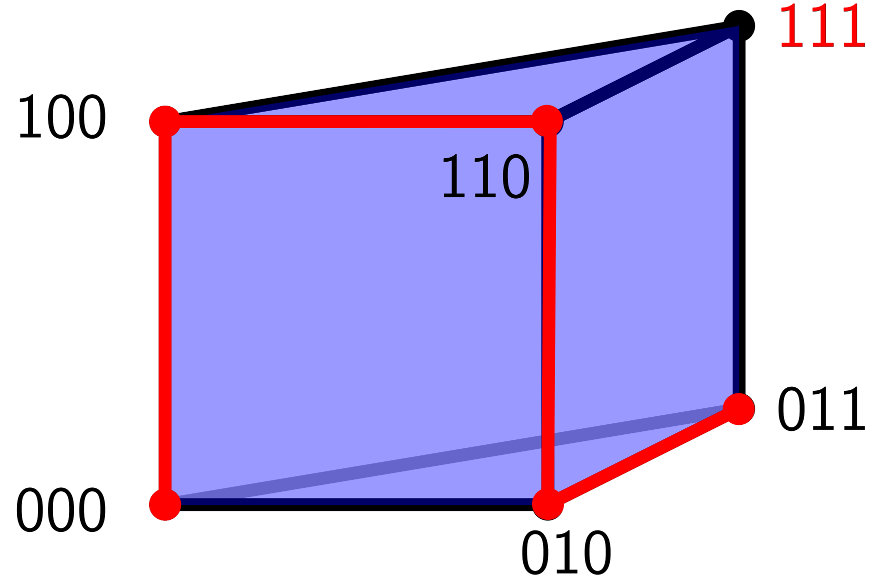

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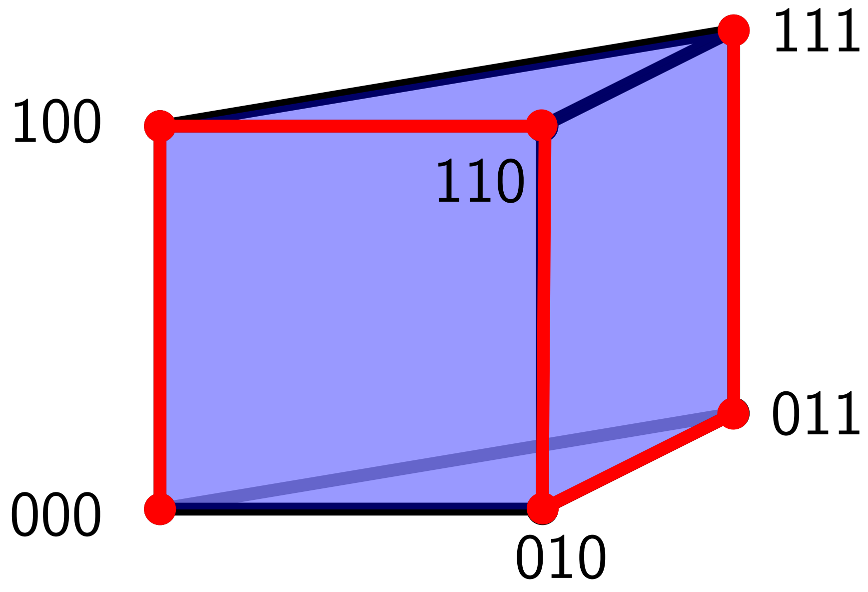

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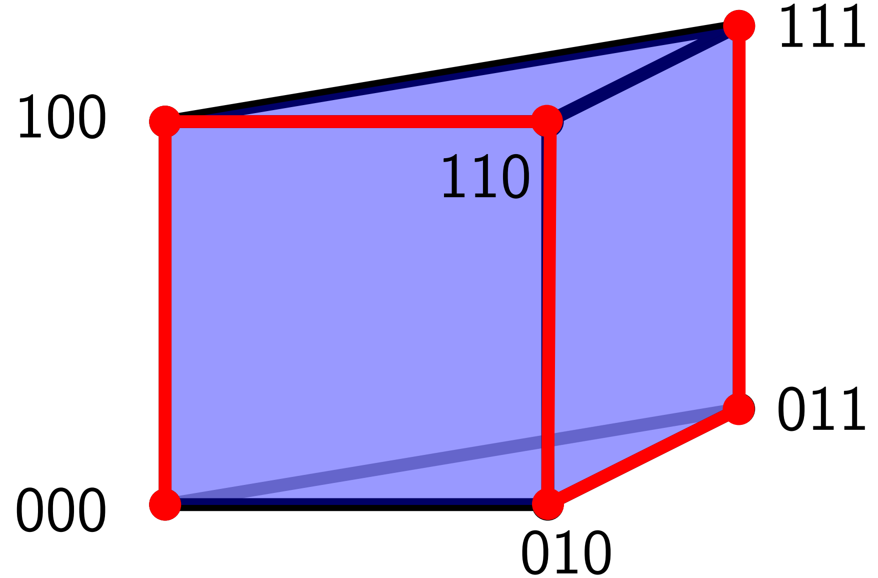

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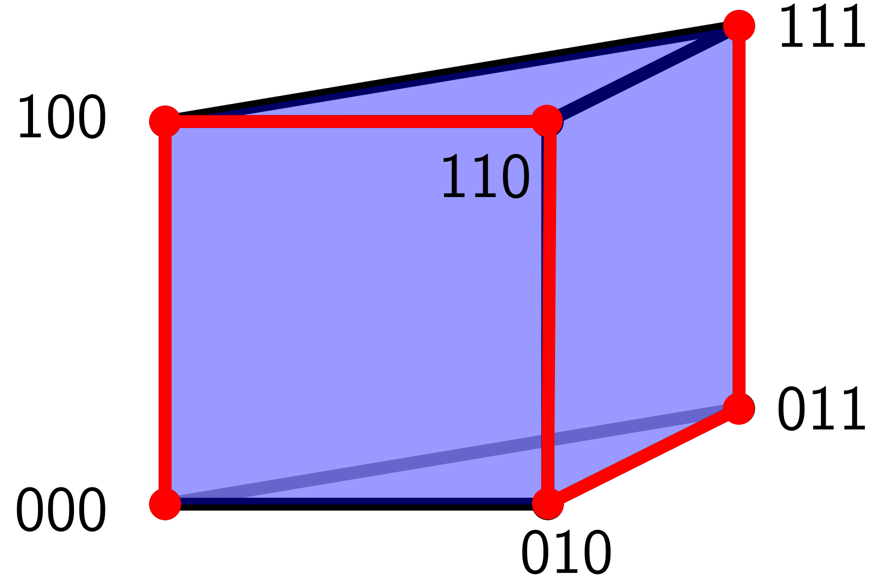

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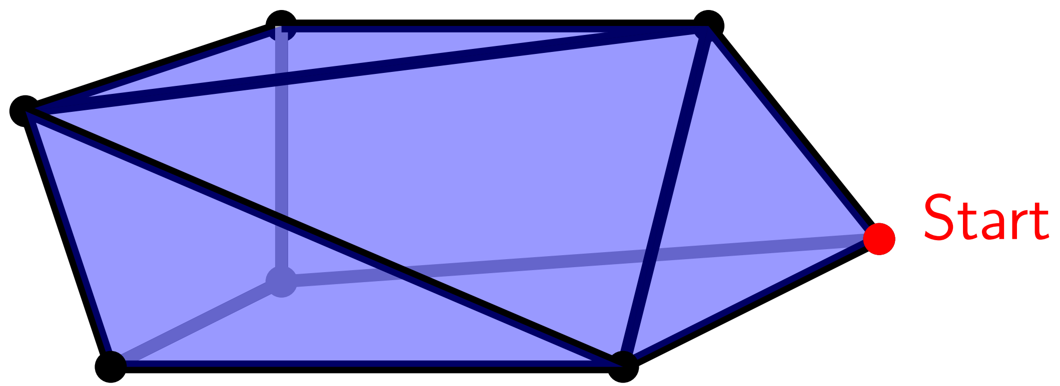

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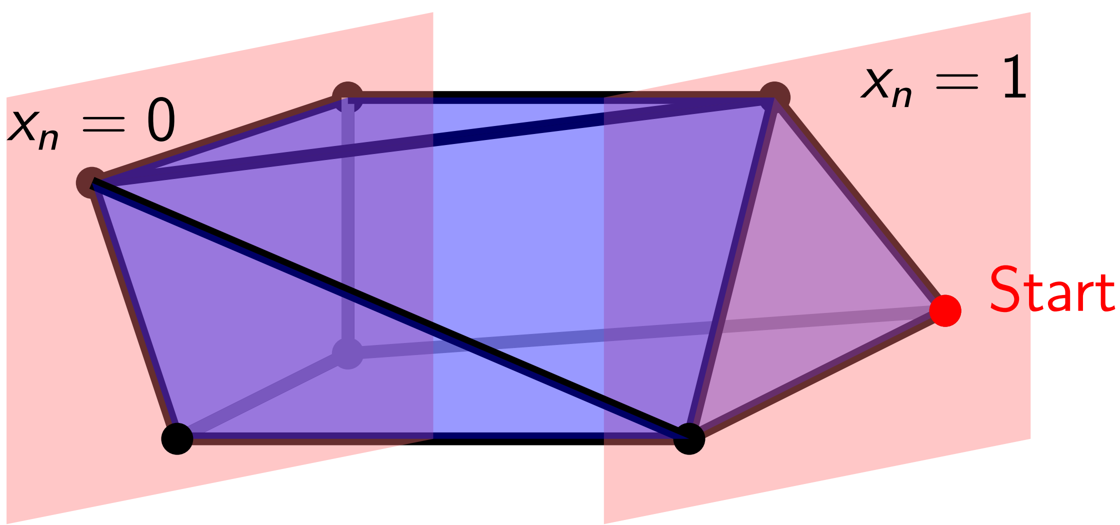

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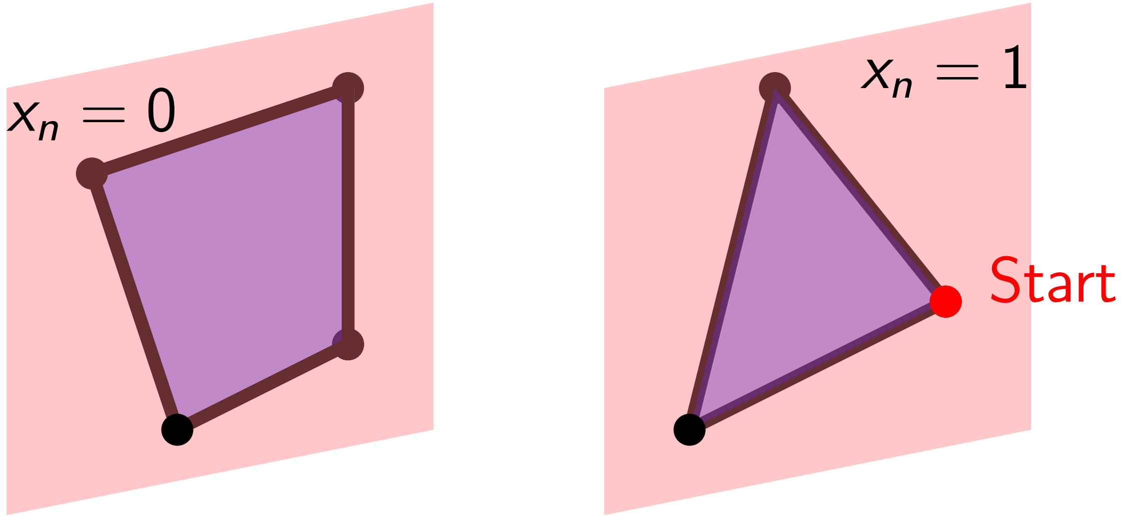

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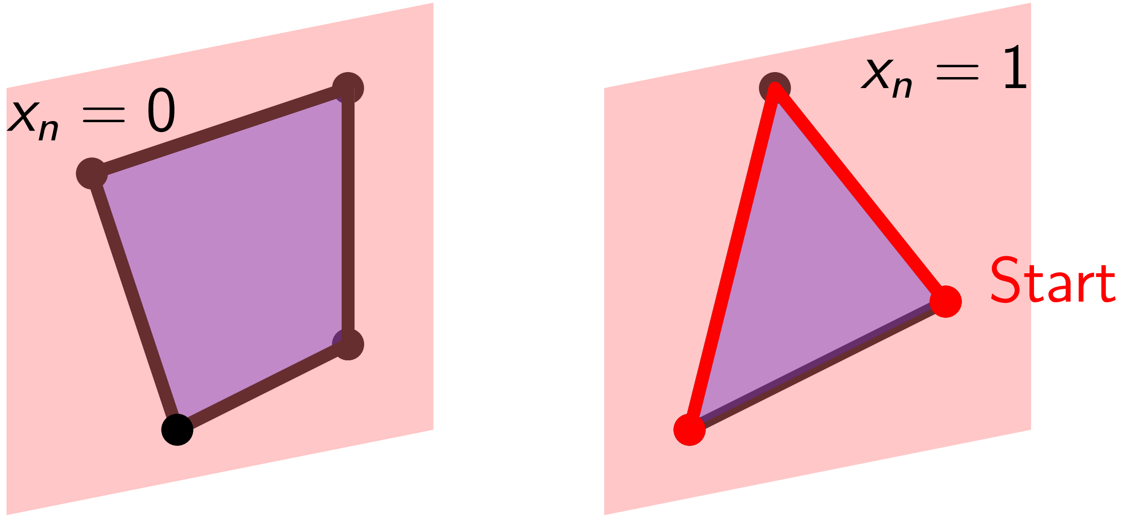

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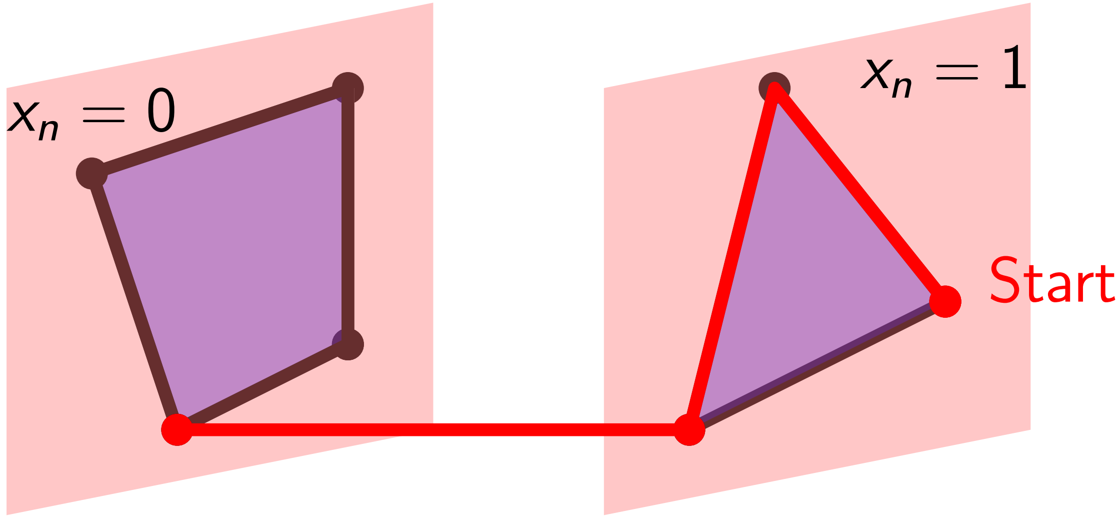

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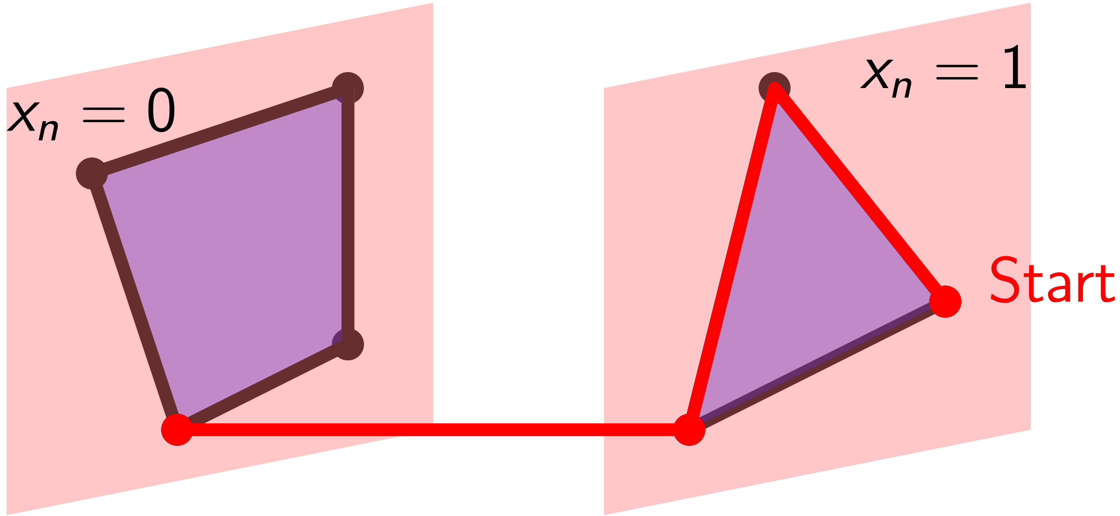

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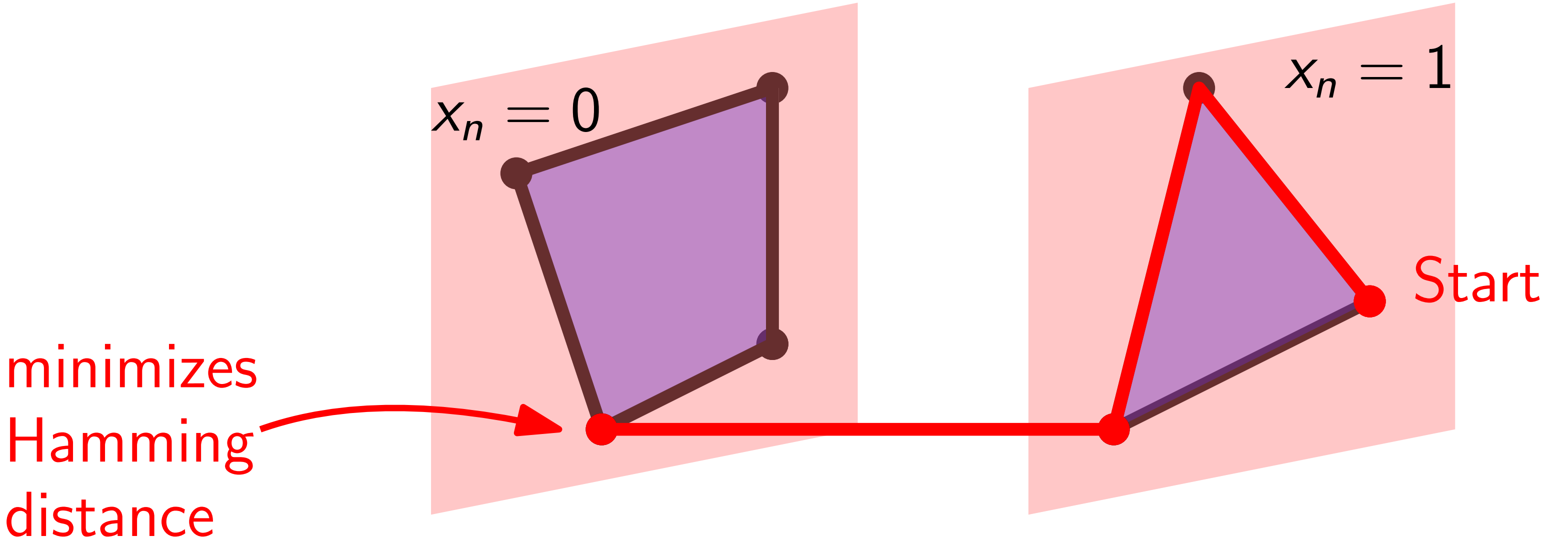

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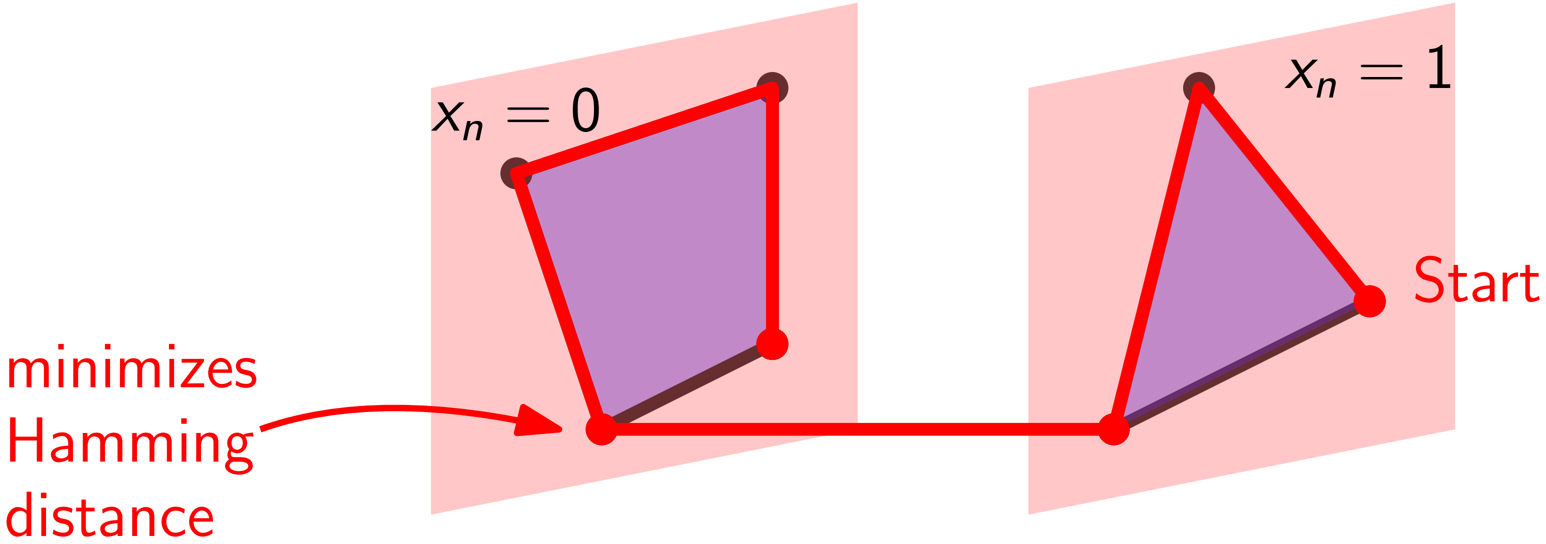

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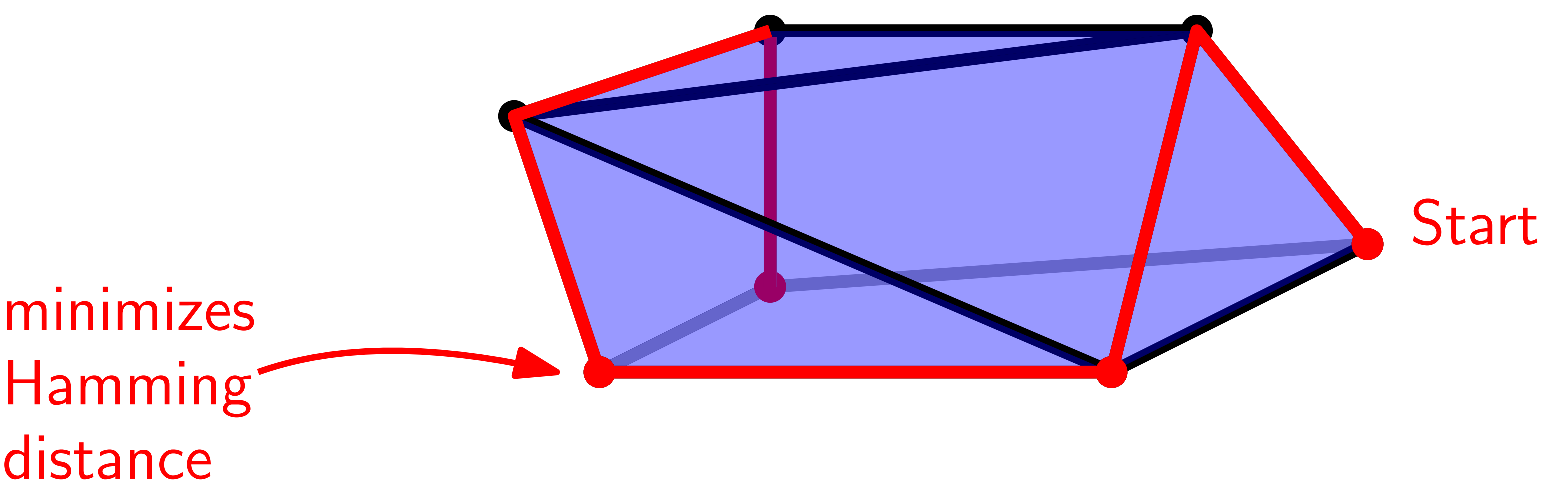

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x_1 & 1 & 1 & 1 & 0 & 0 \\
x_2 & 1 & 0 & 1 & 1 & 0 \\
x_3 & 0 & 1 & 1 & 1 & 0 \\
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x_5 & 1 & 1 & 0 & 0 & 1 \\
x_6 & 0 & 1 & 1 & 0 & 1 \\
x_7 & 1 & 0 & 1 & 0 & 1 \\
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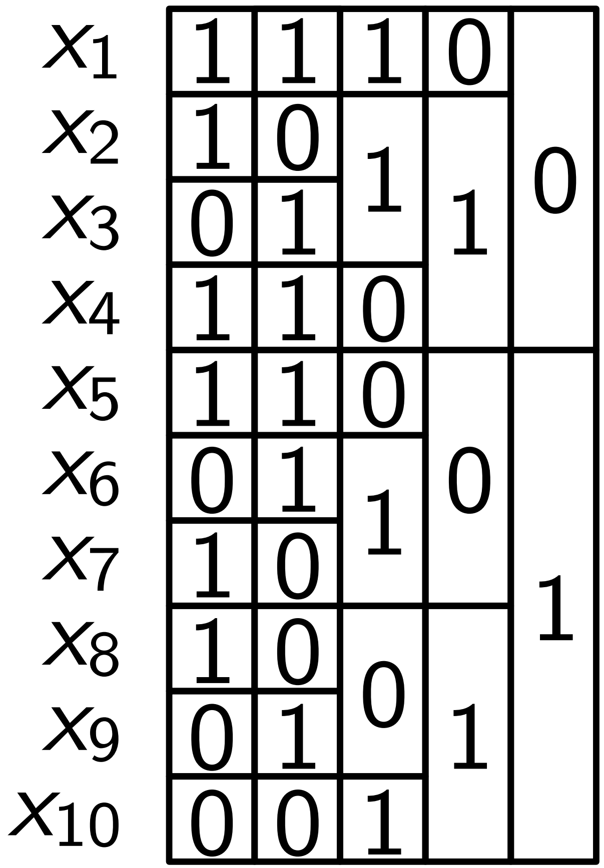

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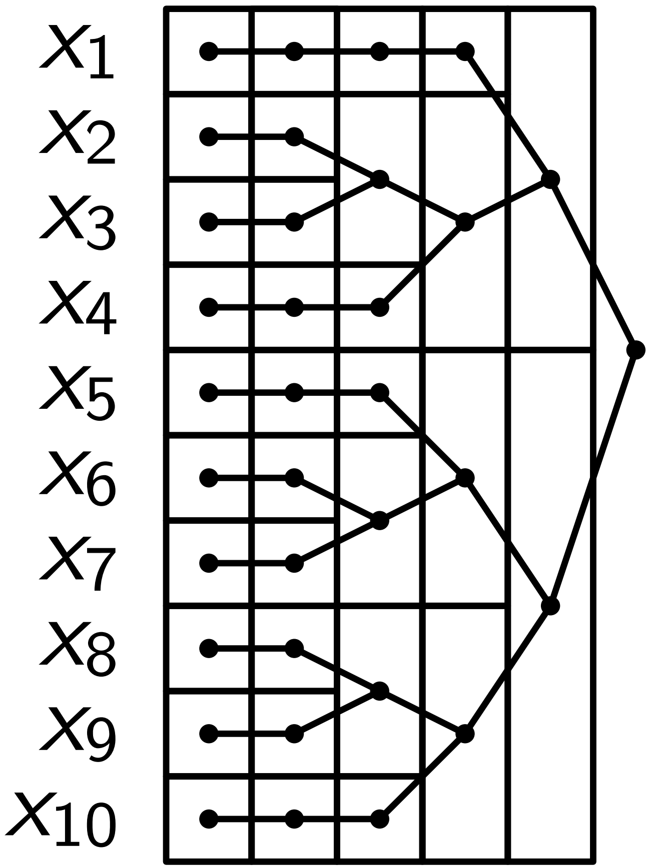

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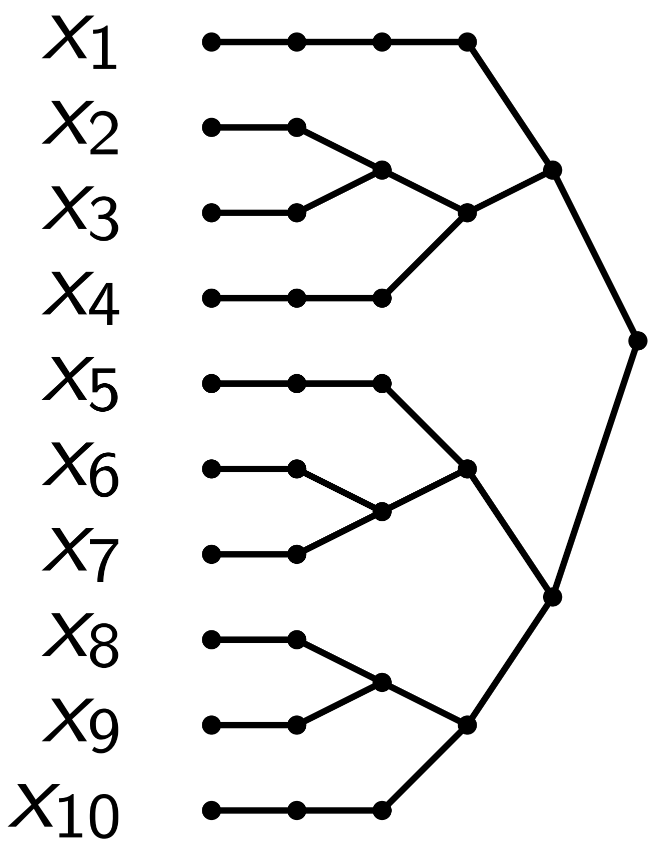

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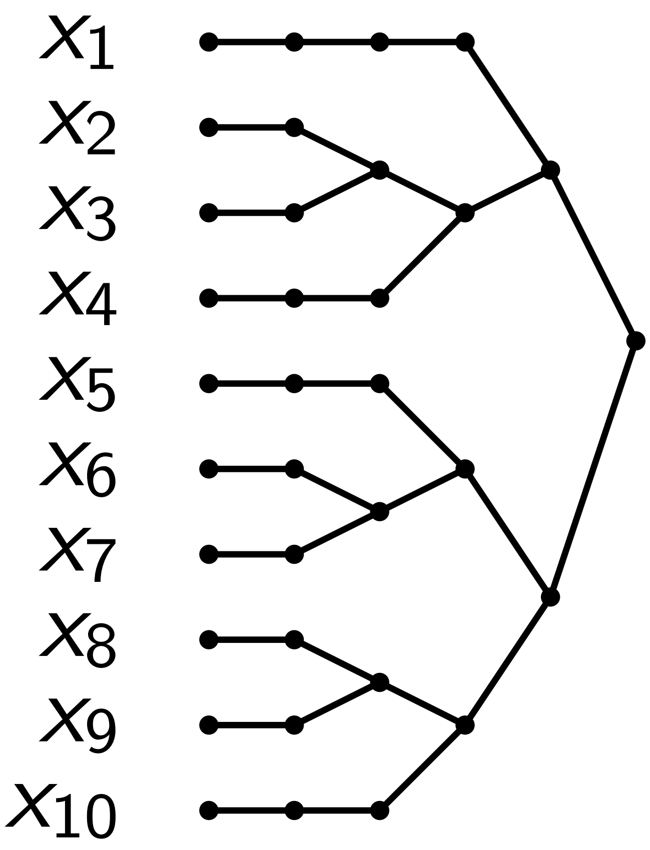

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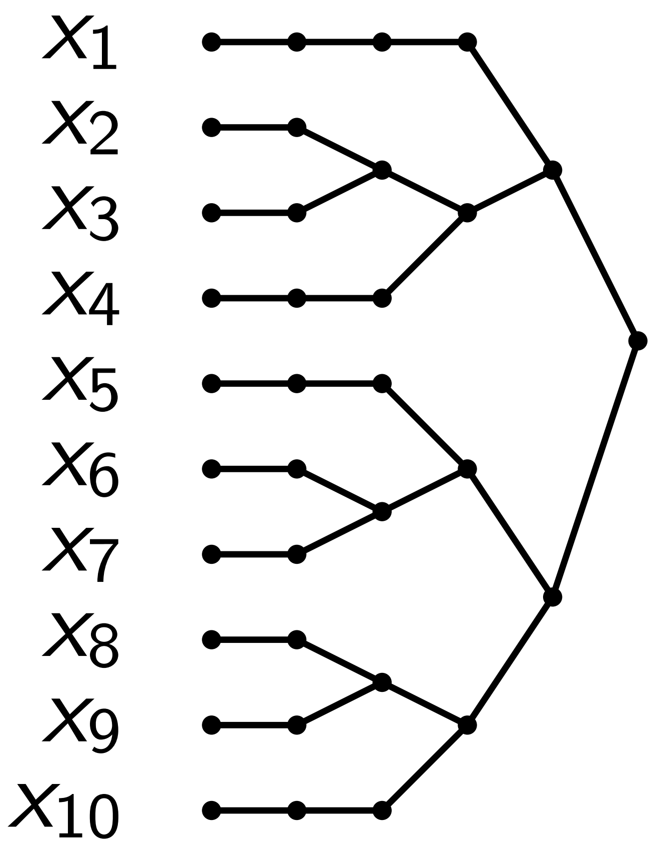

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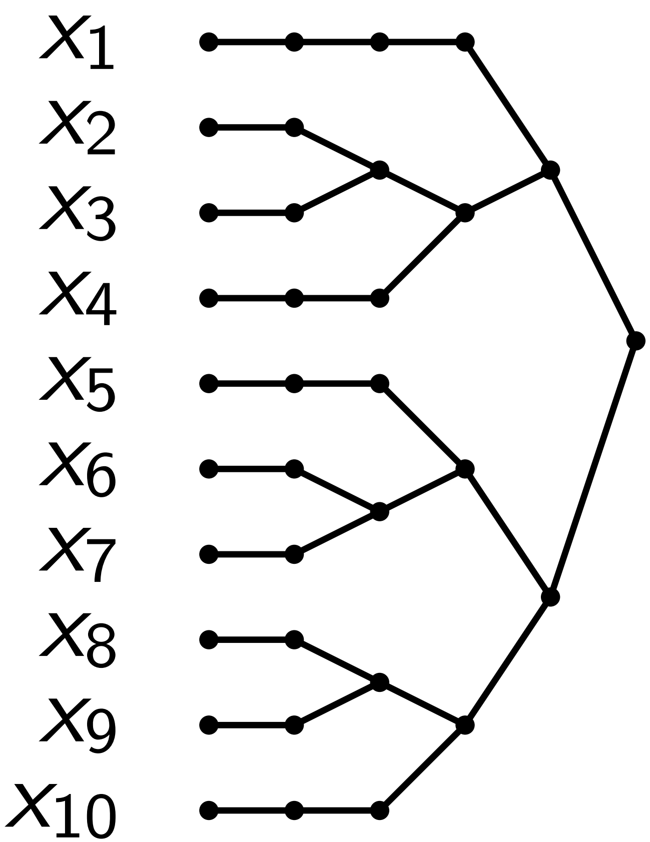

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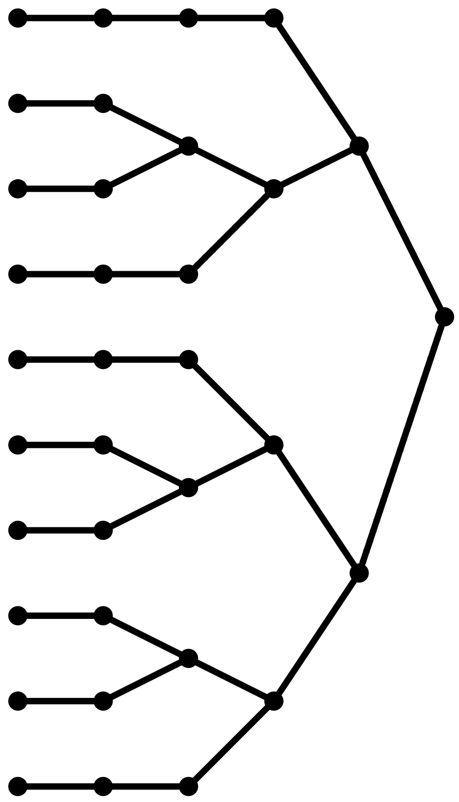
x



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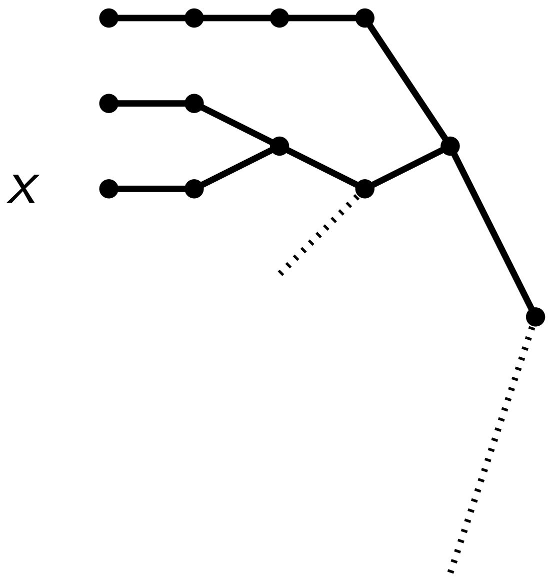

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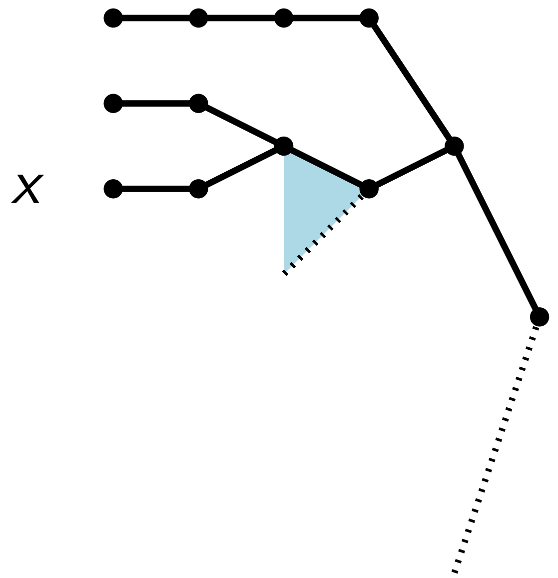

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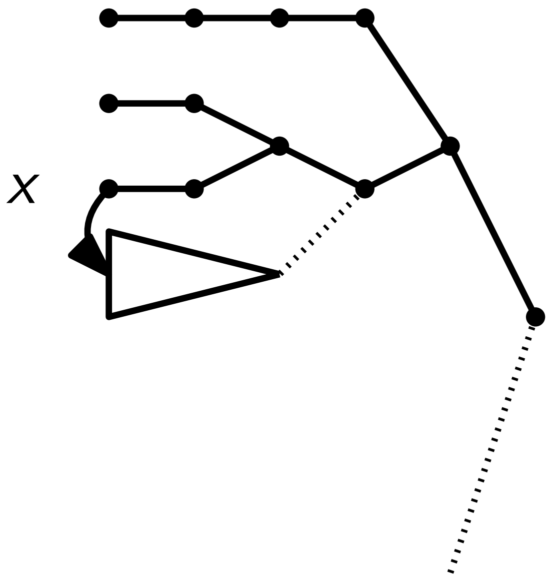

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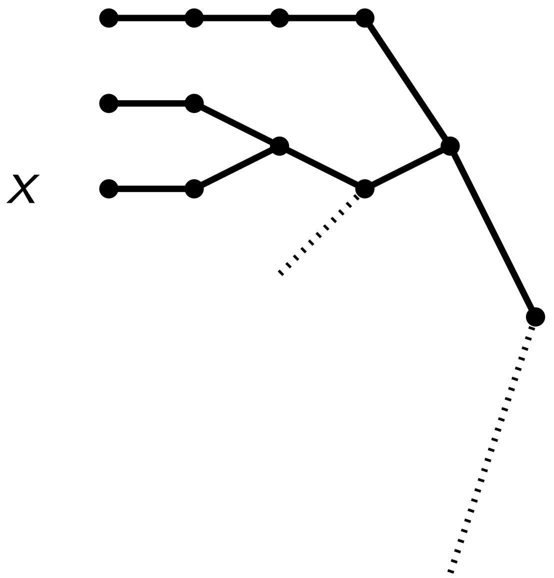

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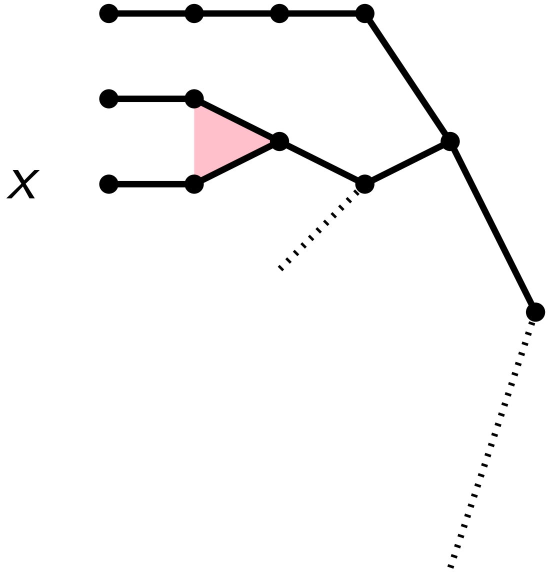

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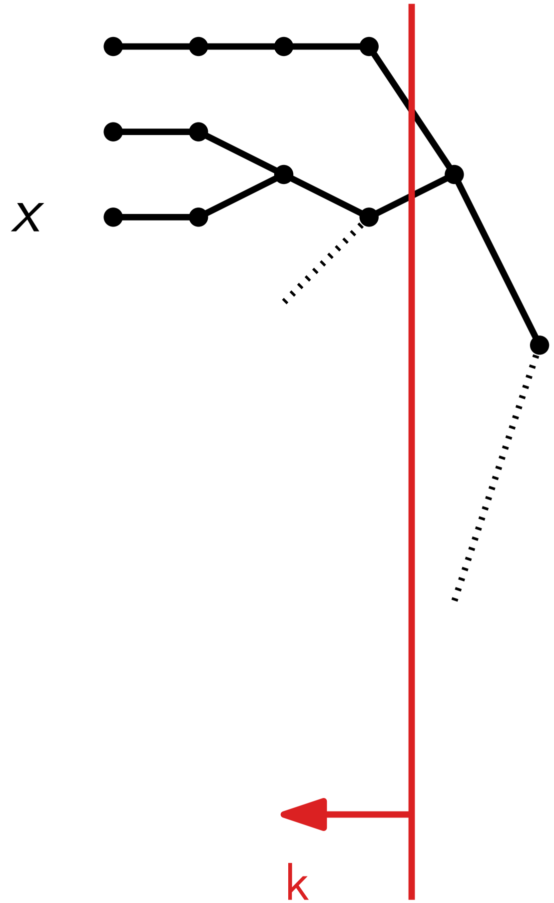

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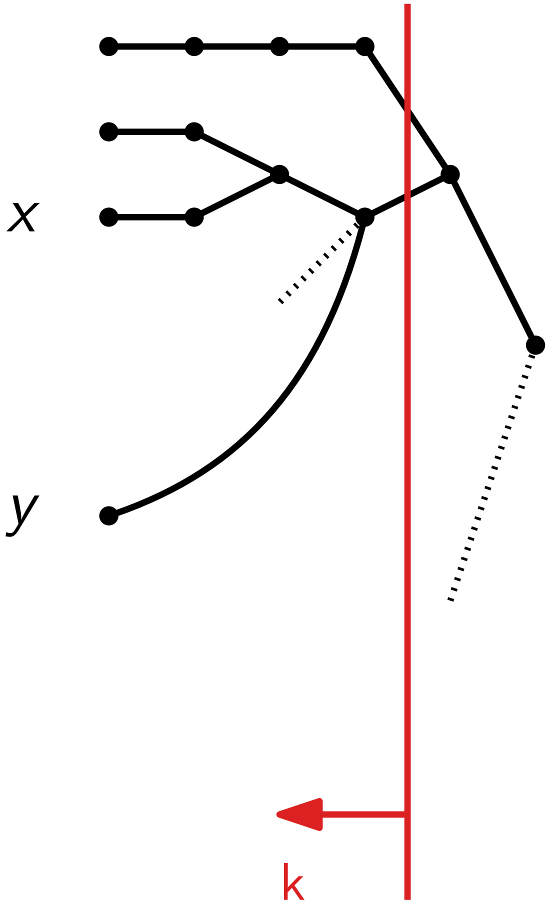

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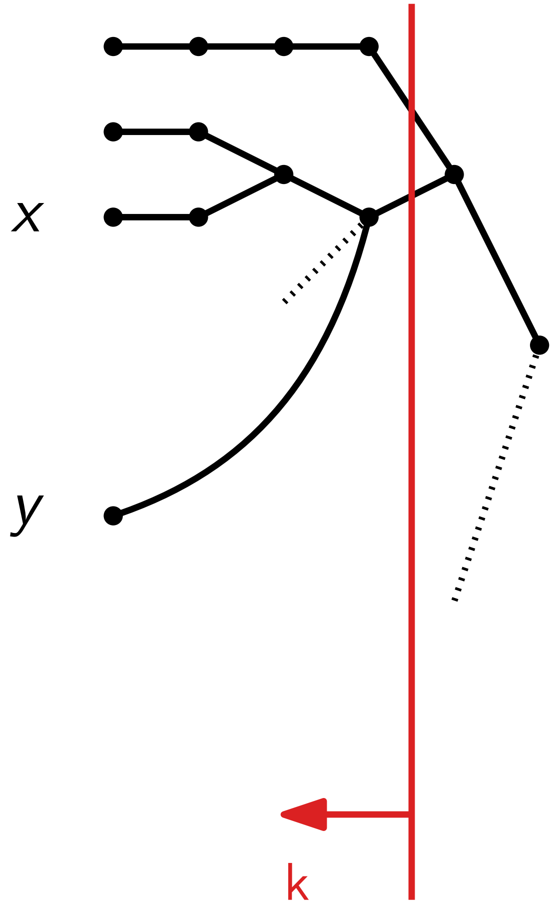

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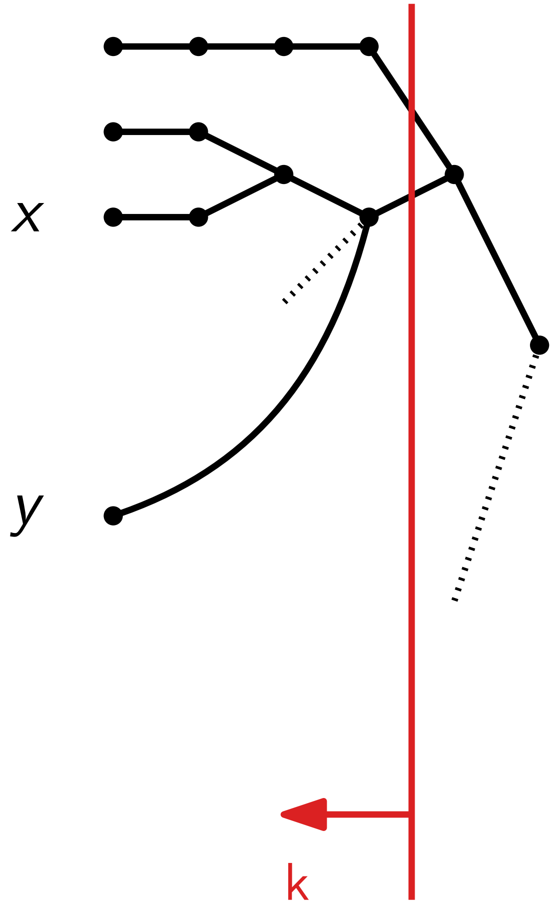

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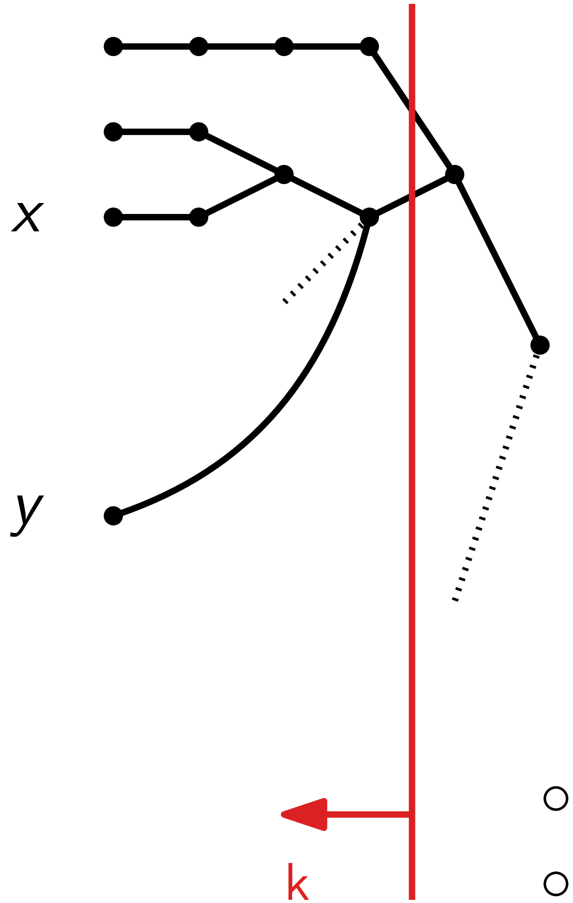

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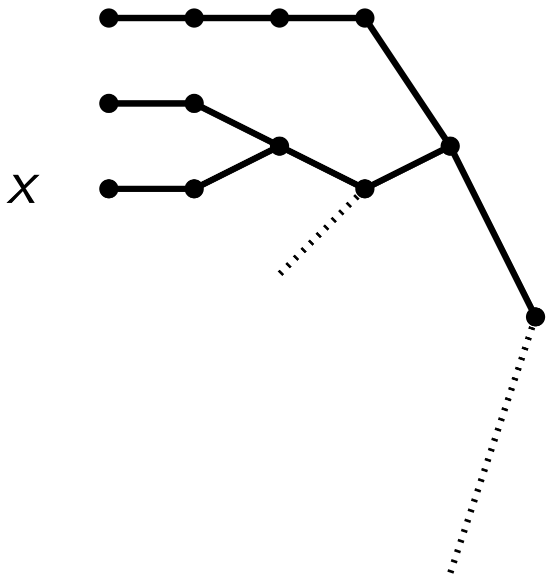

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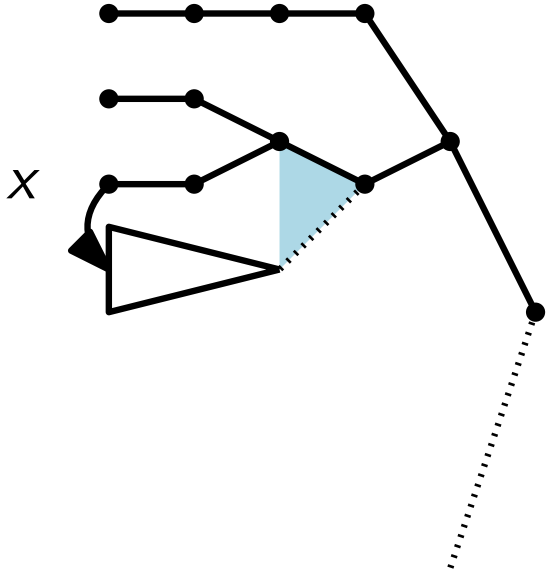

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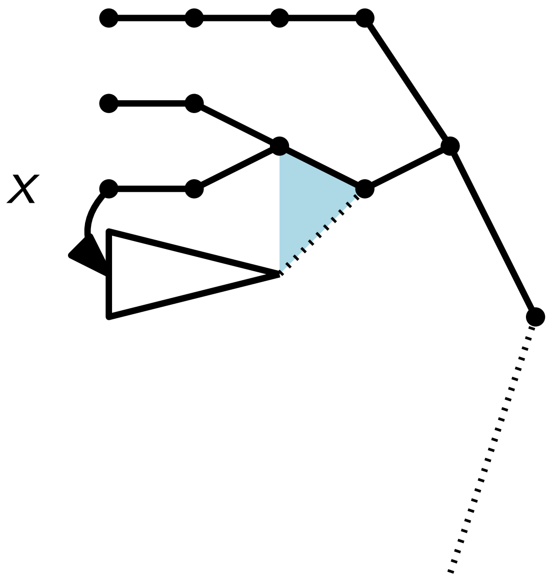

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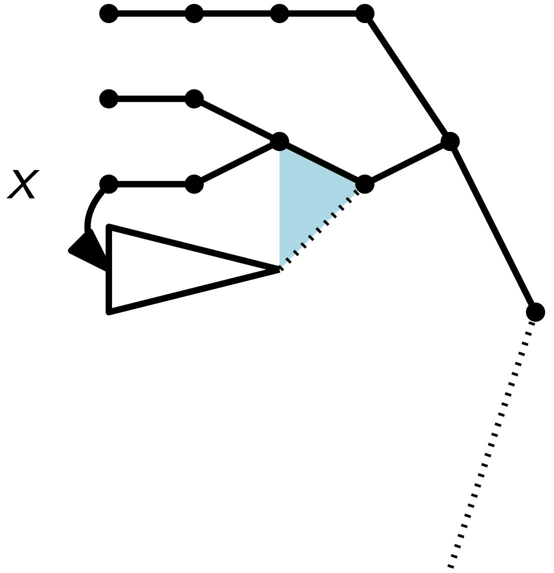

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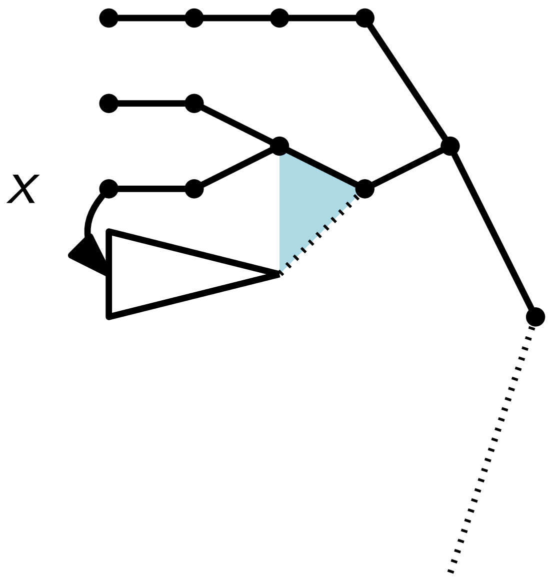

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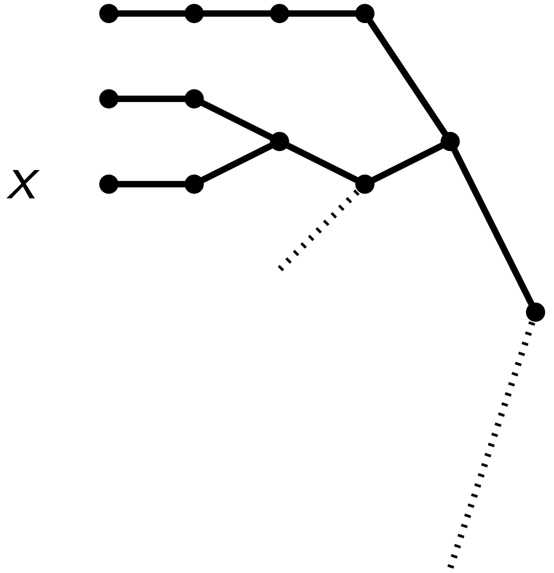

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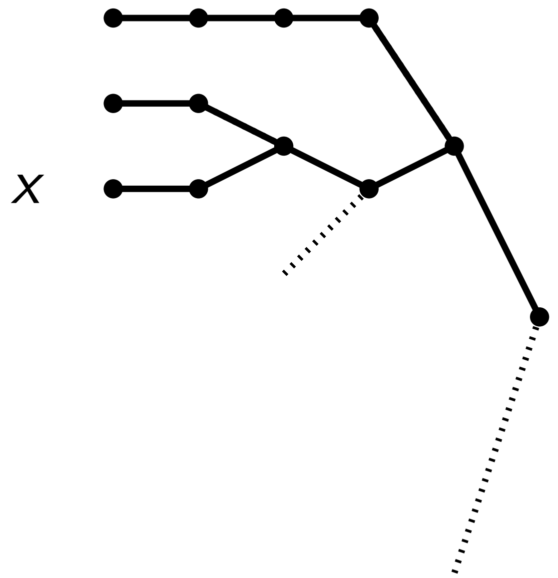

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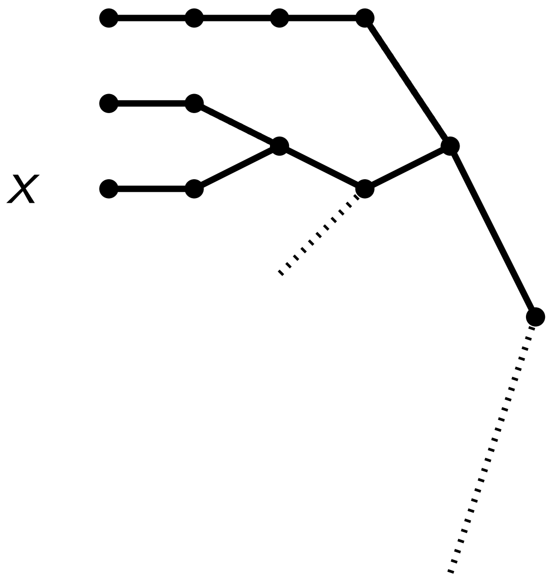

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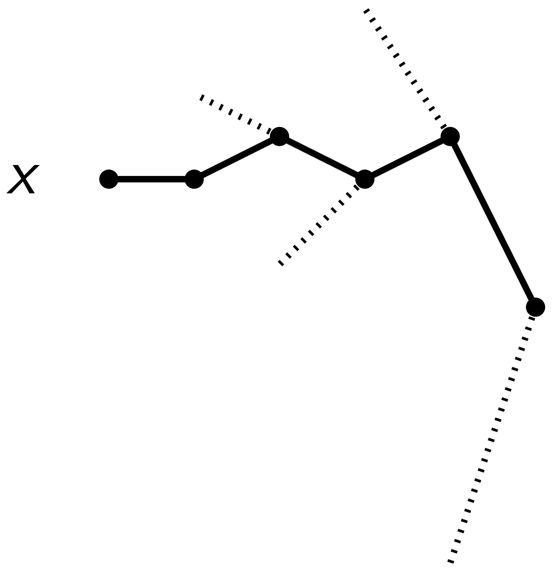

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Thanks for your attention!

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