Realizing the s-Permutahedron via Flow Polytopes

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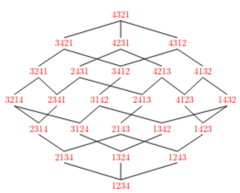
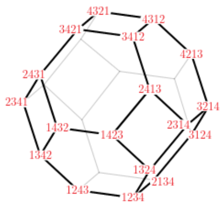
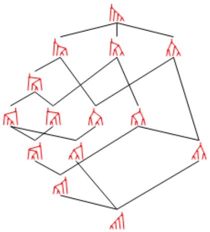
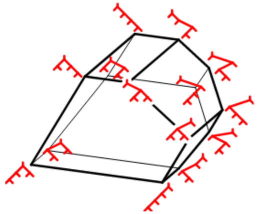
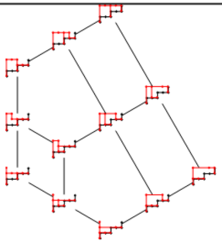
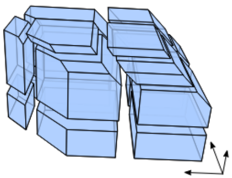
Outline

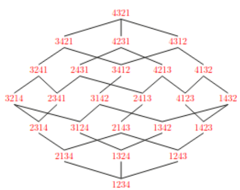
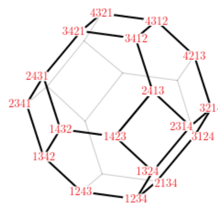
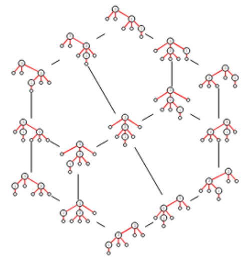
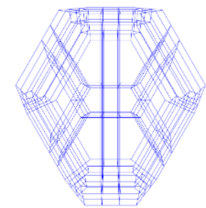
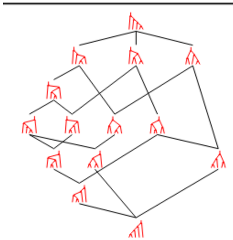
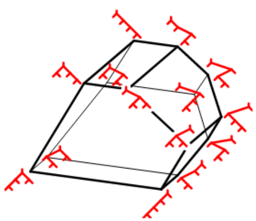
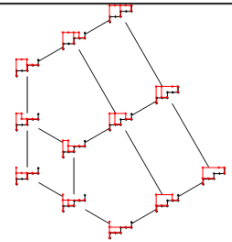
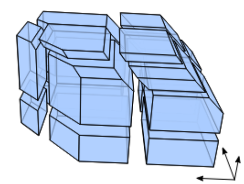
1. *s*-weak order
   - Combinatorial families
   - Flows on graphs
2. Geometric realizations
   - From subdivisions of flow polytope
   - Via the Cayley trick
   - With tropical geometry

Motivation

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

Credit: Pons '19.

Motivation

Weak order	Permutahedron	*s*-Weak order	*s*-Permutahedron
			
			
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Credit: Pons '19.

s-decreasing trees (Ceballos-Pons '20)

Let s be a weak composition ($s_i \geq 0$).

An *s-decreasing tree* is a planar rooted tree on n internal vertices, labeled on $[n]$.

Each vertex labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$.

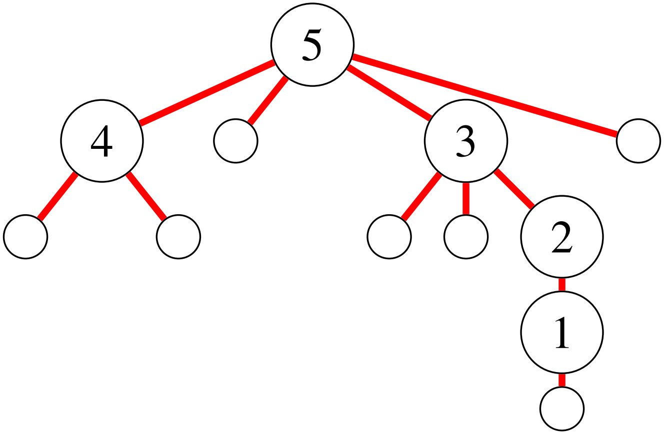
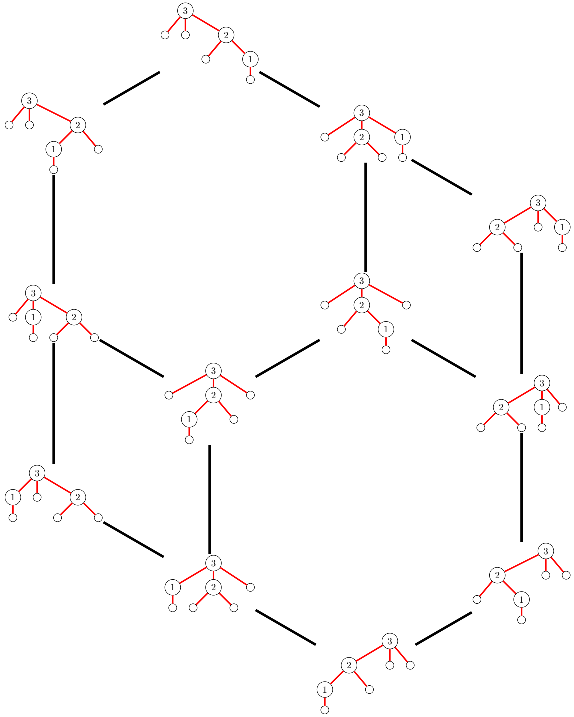


Figure 1: A $(0, 0, 2, 1, 3)$-decreasing tree.

If R, T are s-decreasing trees, the *s-weak order* $\trianglelefteq$ is given by $R \trianglelefteq T$ iff $\mathrm{inv}(R) \subseteq \mathrm{inv}(T)$.



Credit: Ceballos, Pons '19.

Figure 2: The lattice of $(0, 1, 2)$-decreasing trees.

s-Stirling permutations

Let s be a composition ($s_i > 0$). s-decreasing trees are associated to permutations of $1^{s_1} \cdots n^{s_n}$ called *s-Stirling permutations* (also called *121-avoiding s-permutations*).

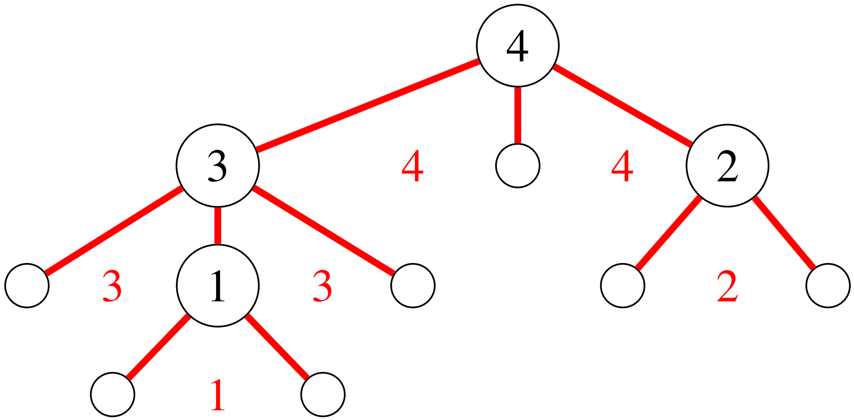
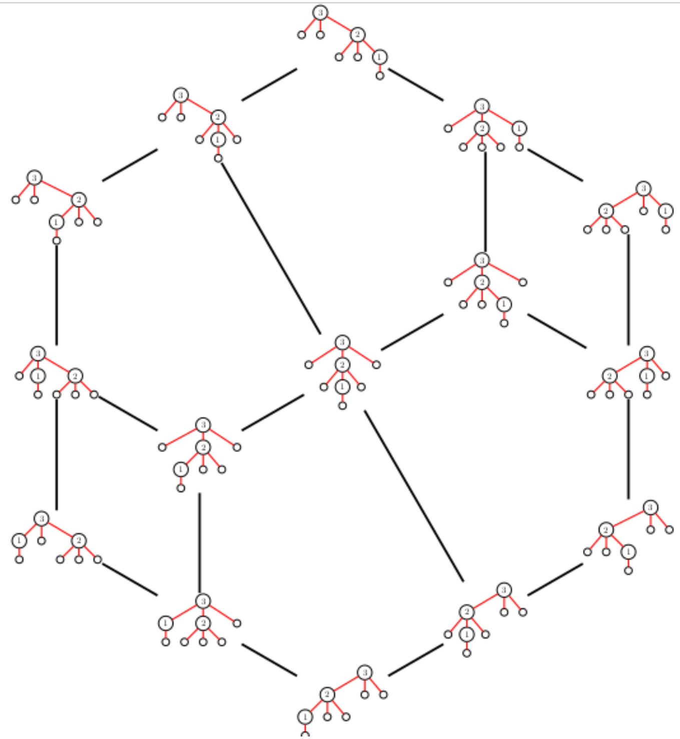


Figure 3: A $(1, 1, 2, 2)$-decreasing tree corresponding to 313442.

s-weak order



Credit: Ceballos-Pons '19.

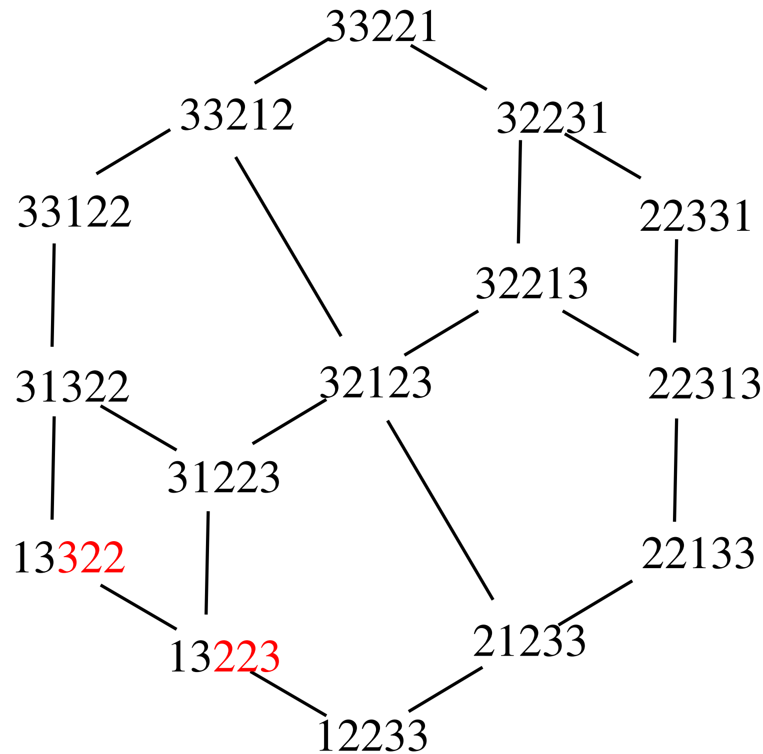
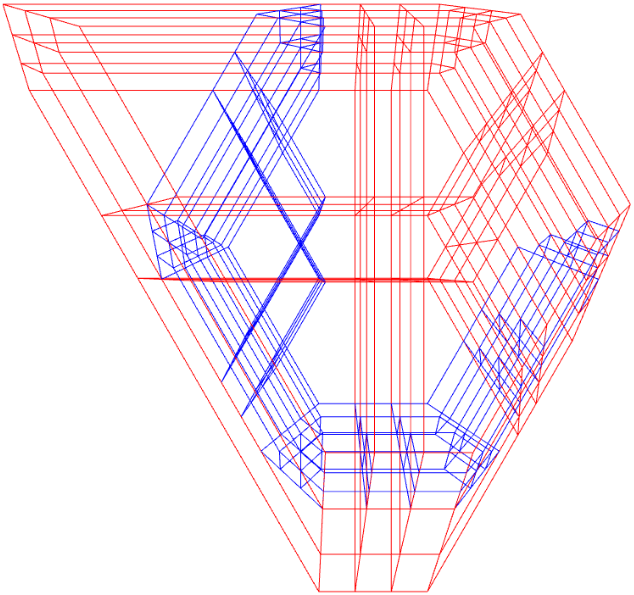


Figure 4: The $(1, 2, 2)$-weak order.

Conjecture 1 (Ceballos-Pons '19)

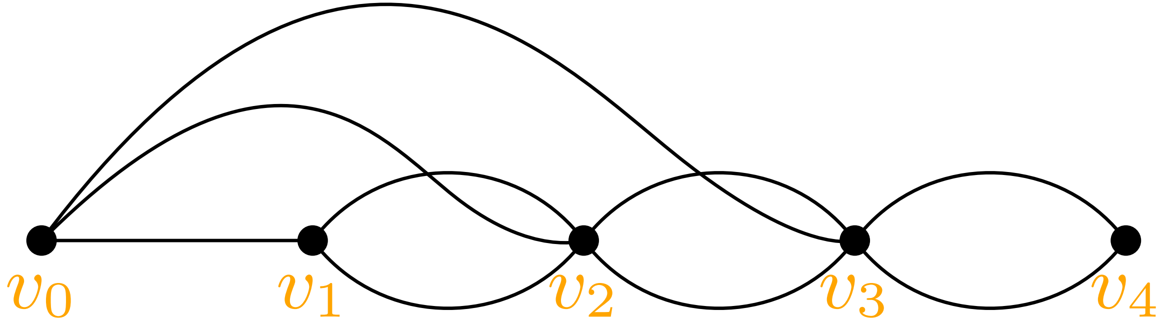
The *s*-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to a zonotope.



Credit: Ceballos-Pons '19.

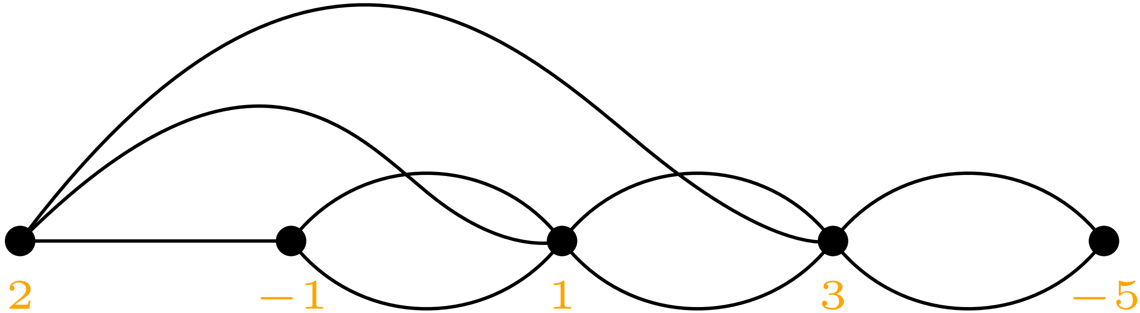
Flows on graphs

Take a digraph G on vertices $\{v_0, \dots, v_n\}$.



Flows on graphs

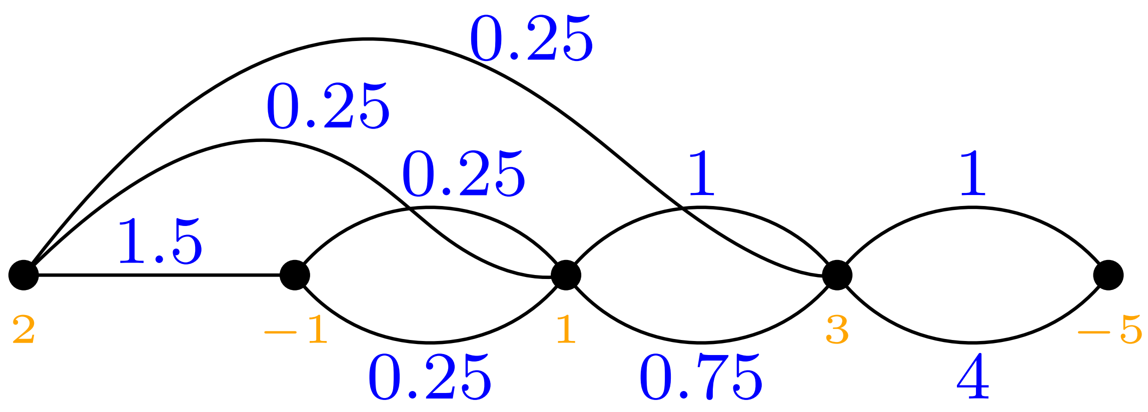
Associate to the vertices a *netflow* $\mathbf{a} = (a_0, \dots, a_n)$ where $a_0 \geq 0$ and $a_n = -\sum a_i$.



Flows on graphs

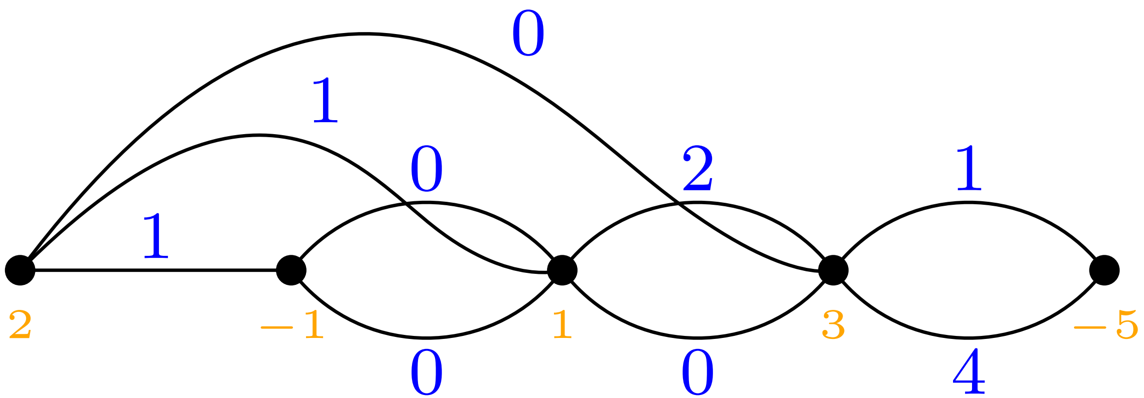
An *admissible flow* of G with netflow **a** is a labelling of the edges $f \in \mathbb{R}^E_{\geq}0$ such that

$$a_i + \sum_{e \in \mathrm{in}(i)} f_e = \sum_{e \in \mathrm{out}(i)} f_e$$



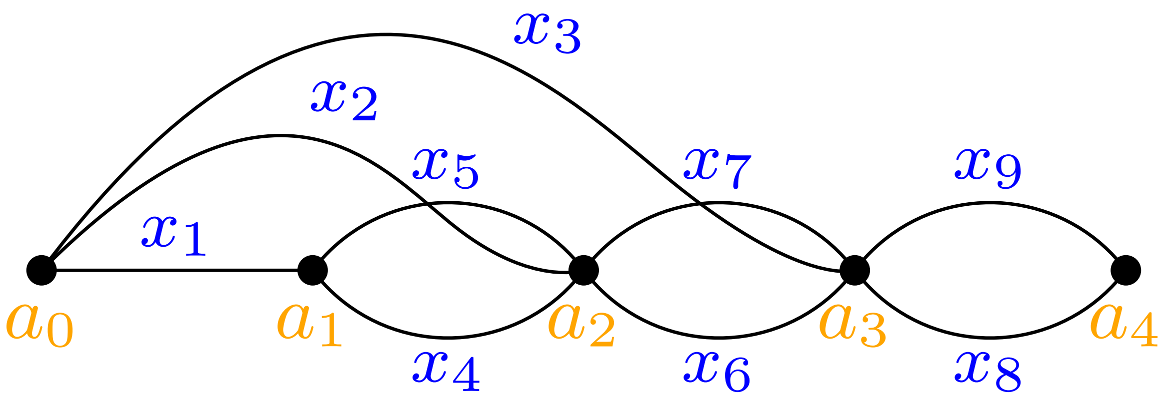
Flows on graphs

An *integer flow* of G is a such a labeling of the edges.



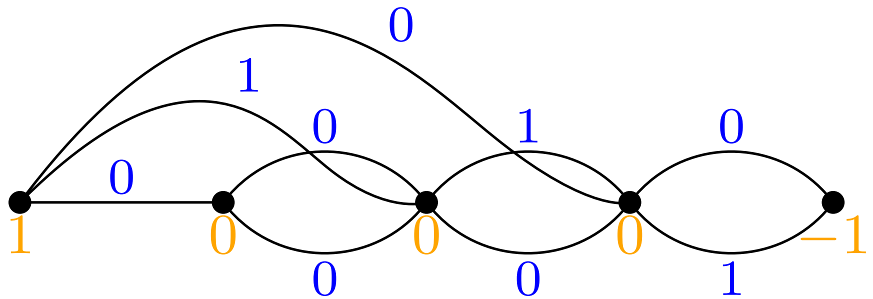
Flows on graphs

The flow polytope $\mathcal{F}_G(\mathbf{a})$ is the convex hull of all admissible flows with netflow $\mathbf{a}$.



Why flow polytope?

- Although they live in $\mathbb{R}^E$ they have dimension $|E| - |V| + 1$.
- Their integer points are nice. In the case $\mathbf{i} = (1, 0, \ldots, 0, -1)$, the vertices of $\mathcal{F}_G(\mathbf{i})$ are the indicator vectors of the routes of G.

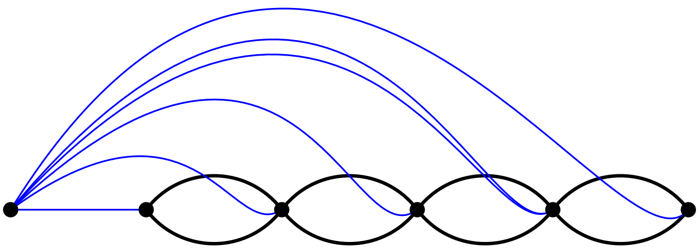


- They have nice triangulations.

Which flow polytope?

Given a composition $s = (s_1, \ldots, s_n)$, G_s is the multi-digraph on $\{v_{-1}, \ldots, v_n\}$ such that there is

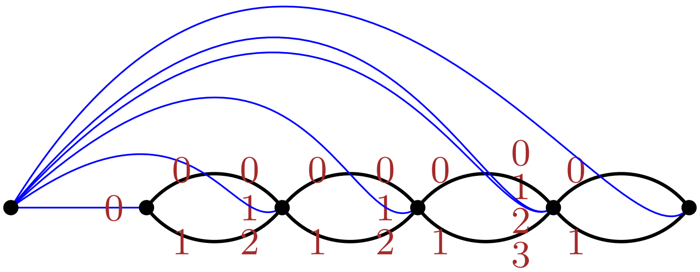
- 1 edge (v_{-1}, v_0),
- 2 edges (v_i, v_{i+1}) for $i \in \{0, \ldots, n-1\}$,
- $s_{n+1-i} - 1$ edges (v_{-1}, v_i).



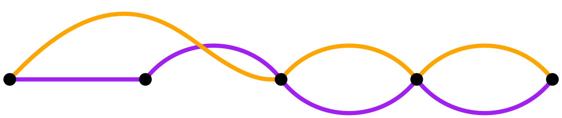
The graph $G_{(2,3,2,2)}$.

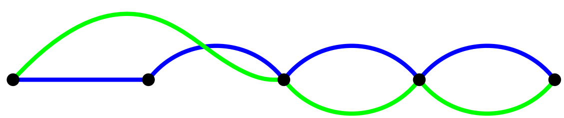
Framings and coherence

A *framing* is a total order on the in-edges and out-edges of each vertex.



A pair of routes are *coherent* if wherever they meet they have the same order of entrance and of exit.



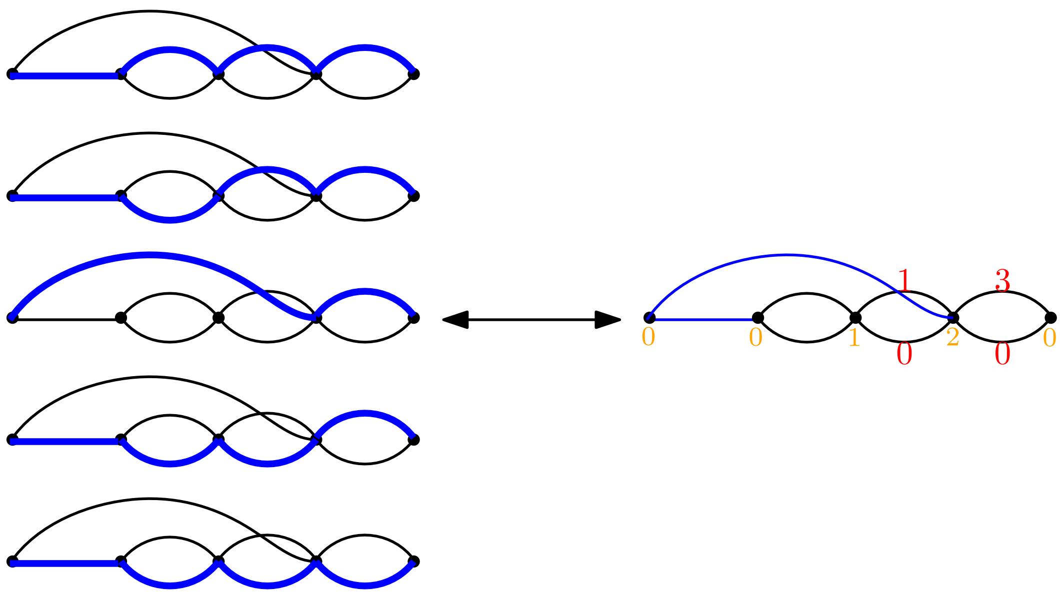


A *maximal clique* C is a maximal set of coherent routes.

Triangulations

Theorem (Mészáros, Morales, Striker, 12')

The maximal cliques of a framed graph $(G, \preceq)$ *are in bijection with the integer flows of* $\mathcal{F}_G(\mathbf{d})$ *where* $d_i = \mathrm{indeg}(v_i) - 1$.

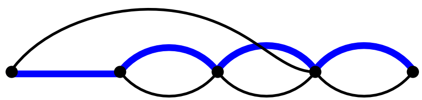


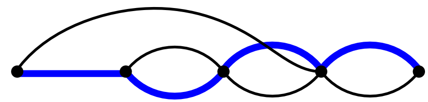
Danilov-Karzanov-Koshevoy Triangulations

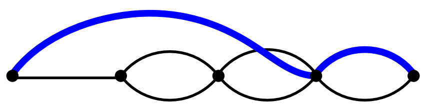
For a clique C, Δ_C denotes the simplex with vertices the indicator vectors of the routes in C.

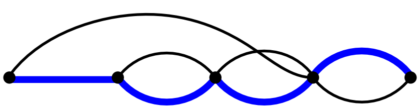
Theorem (DKK, 12)

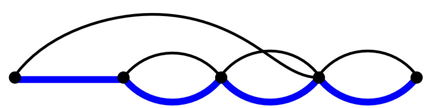
The maximal simplices Δ_C form a regular triangulation of $\mathcal{F}_G(\mathbf{i})$, called the DKK triangulation of $\mathcal{F}_G(\mathbf{i})$ with respect to the framing $\preceq$.

$l_w^0 =$ 

$l_w^1 =$ 

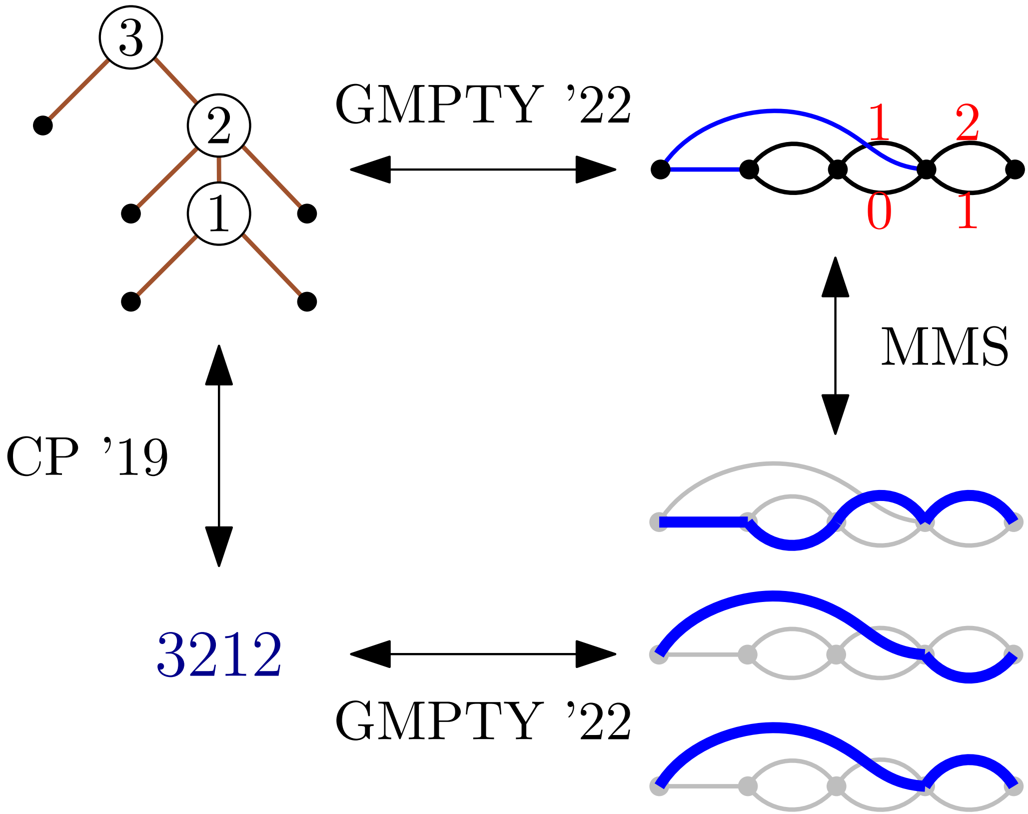
$l_w^2 =$ 

$l_w^3 =$ 

$l_w^4 =$ 

Theorem (GMPTY, 22')

The s-decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_{G_s}, \preceq)$.



Theorem (GMPTY, 22')

Moreover, two simplices are adjacent if and only if there is a cover relation in the s-weak order.

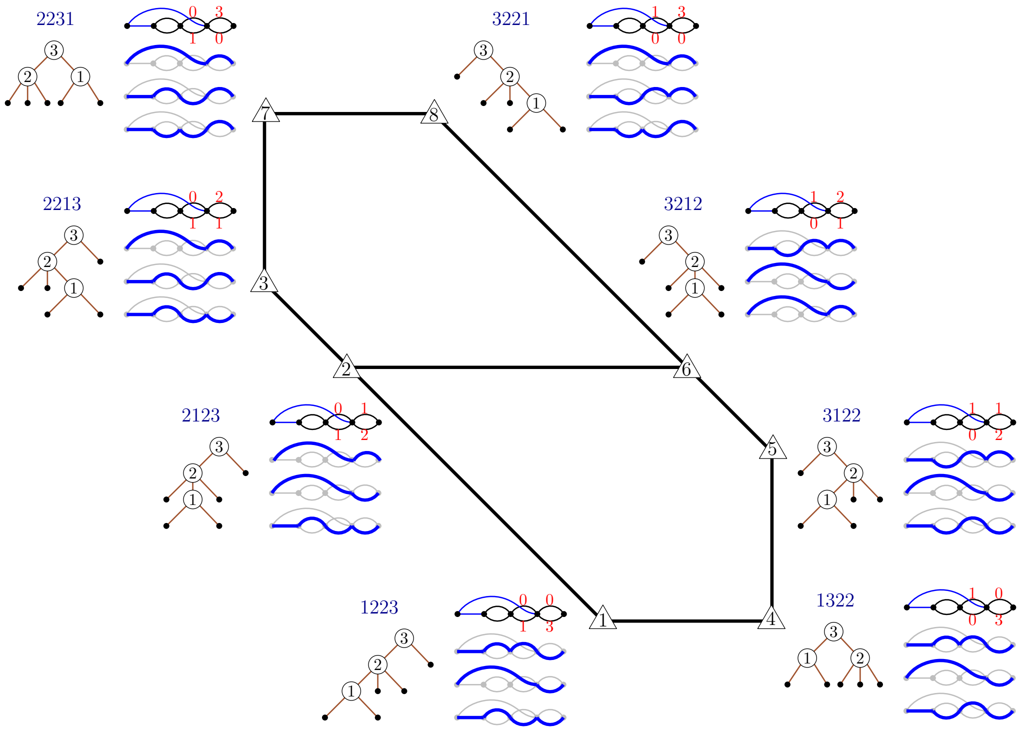


Figure 7: Dual of the DKK triangulation for $s = (1, 2, 1)$.

Problem: This lives in dimension $m - n + 1$ not $n - 1$.

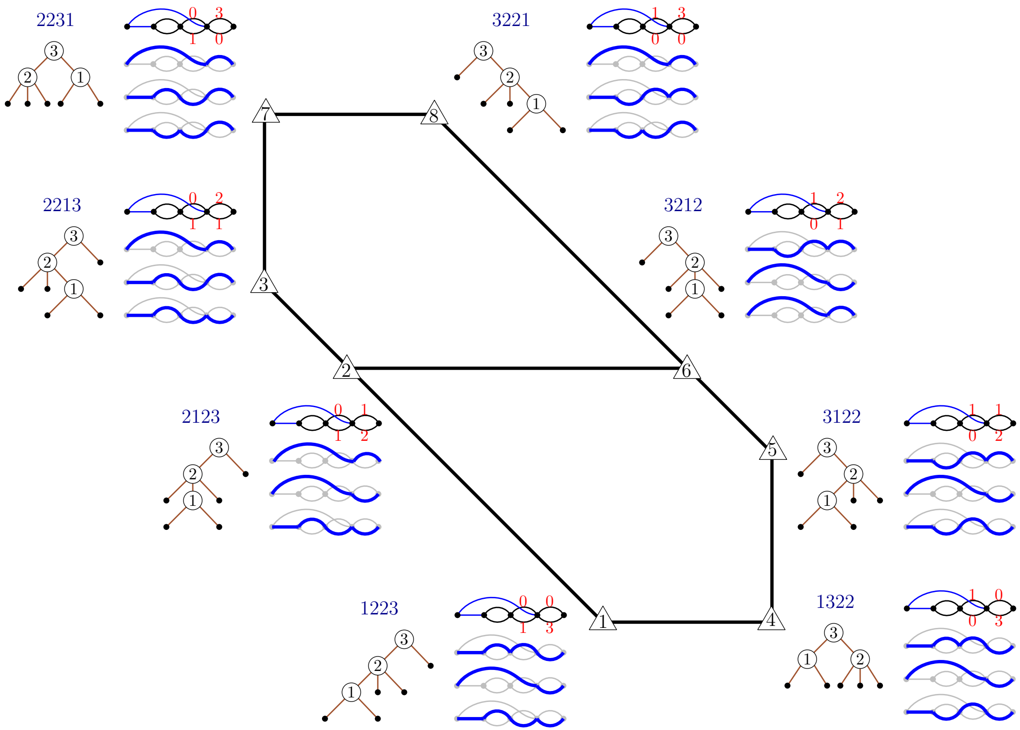
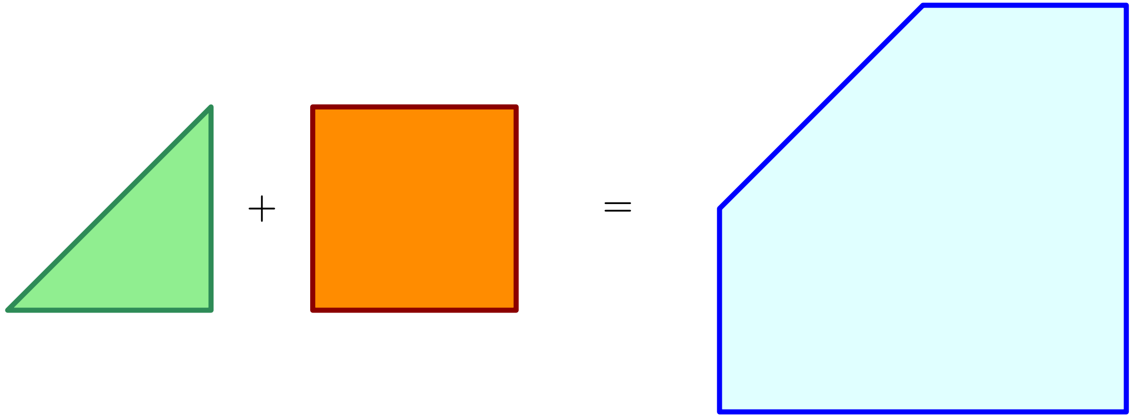


Figure 8: Dual of the DKK triangulation for $s = (1, 2, 1)$.

Minkowski sums

- Given polytopes $P_1, \dots, P_k$ in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P_1 + \ldots + P_k := \{x_1 + \ldots + x_k \mid x_i \in P_i\}$.



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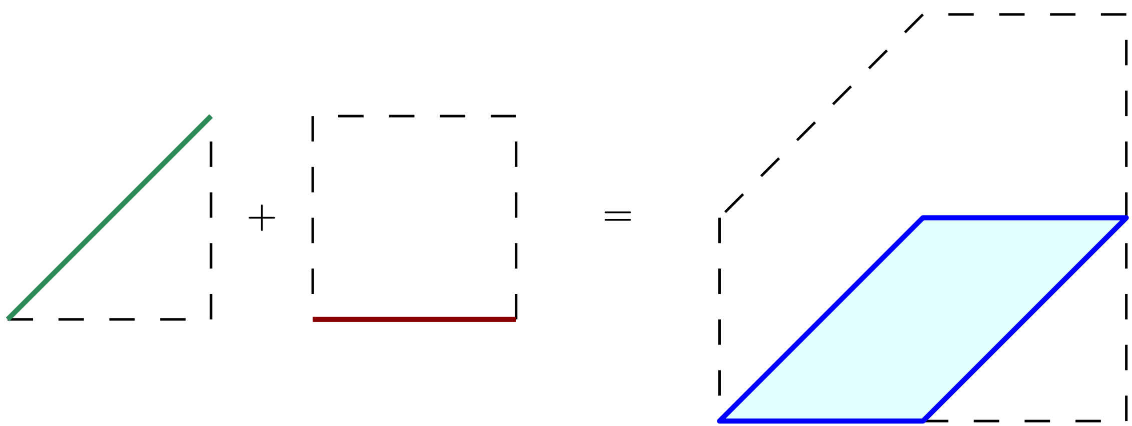
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.



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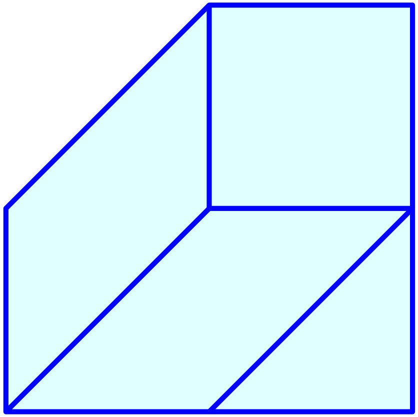
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.



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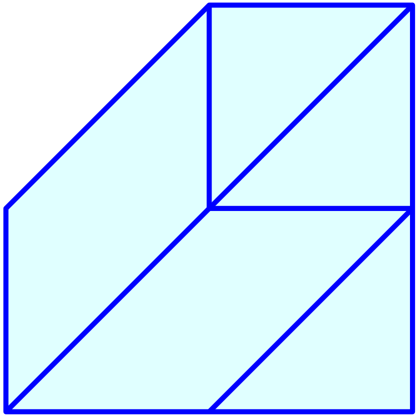
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.
- A *mixed subdivision* is a collection of Minkowski cells whose union covers the Minkowski sum and they intersect properly.
- A *fine mixed subdivision* is a minimal mixed subdivision via containment.

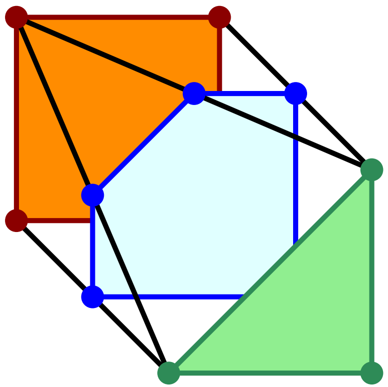


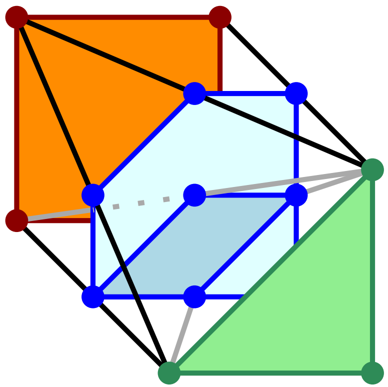
Cayley Trick

$\mathcal{C}(P_1, \ldots, P_k) := \text{conv}(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ is the *Cayley embedding* of $P_1, \ldots, P_k$.

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \ldots, P_k)$ are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \ldots + P_k$.





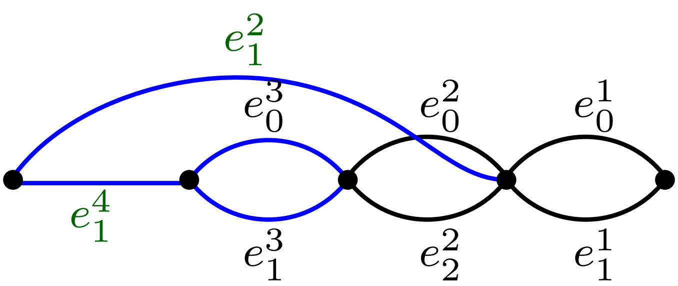
Flow polytopes are Cayley embeddings

Theorem (GMPTY, 22')

The s-decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ *given by*

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1} (s_i - 1)\square_{i-1}.$$

Proof : The flow polytope of G_s is a Cayley embedding of hypercubes.



$s = (1, 2, 1)$

Mixed subdivision of hypercubes

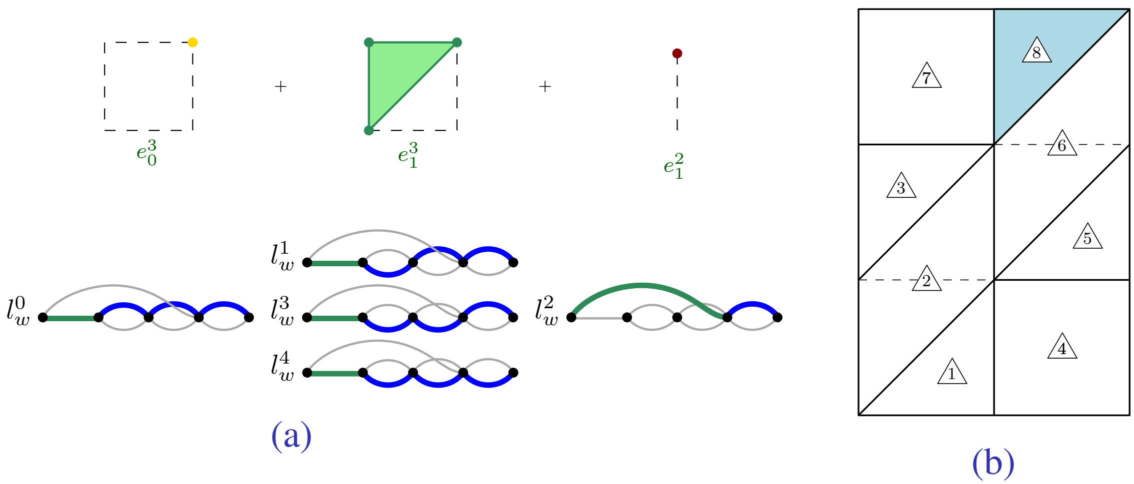
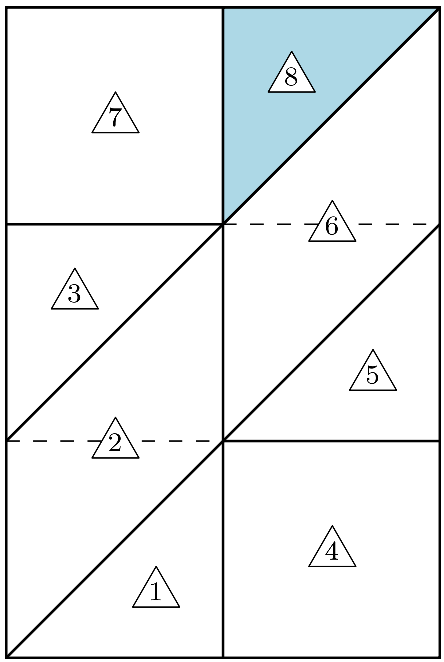
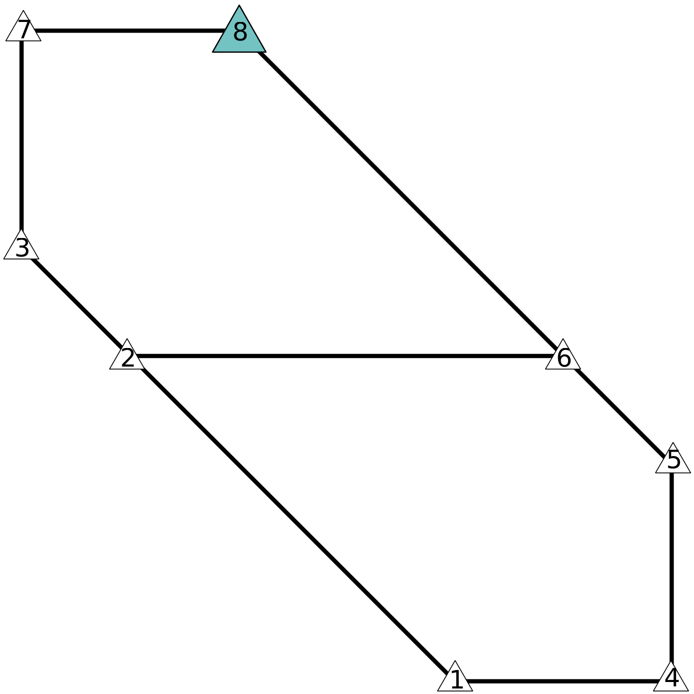


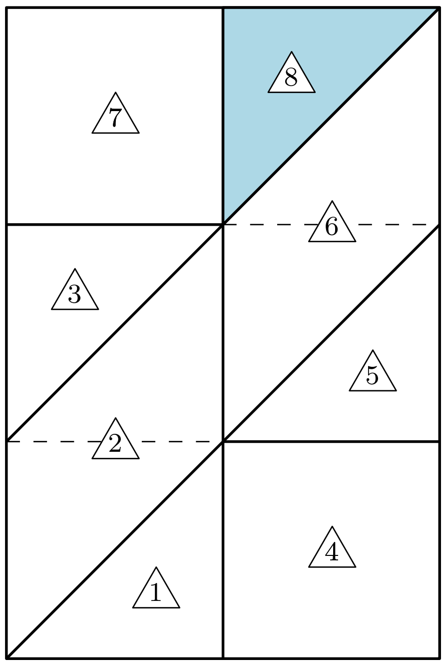
Figure 9: (a) Summands of the Minkowski cell corresponding to $w = 3221$. (b) Mixed subdivision of $2\square_2 + \square_1$ leading to the $(1, 2, 1)$-permutahedron.

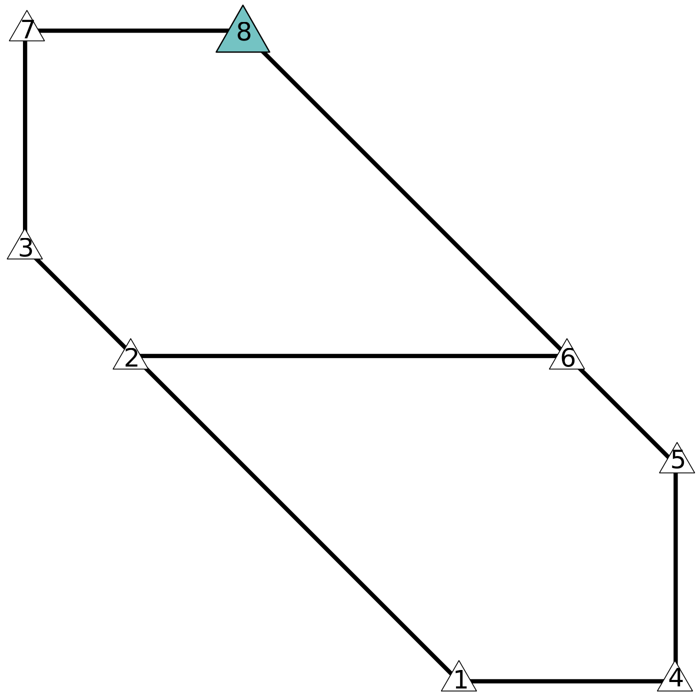
From the mixed subdivision to a dual polyhedral complex





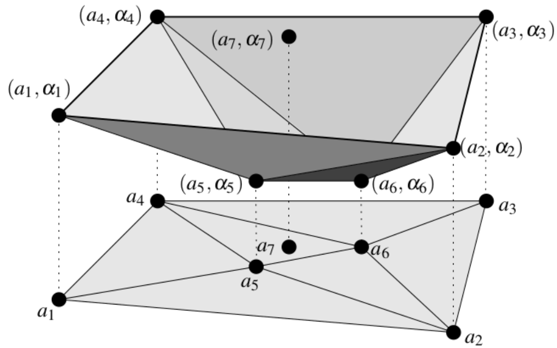
Problem: This dual does not give explicit coordinates.





Tropicalizing triangulations

A regular subdivision $\mathcal{S}$ of a point configuration $\mathcal{A}$ can be obtained as the lower faces of the points of $\mathcal{A}$ lifted by an admissible height function h.



Credit: Rambau '96

Such lifted configuration corresponds to the *tropical polynomial* (in the min-plus algebra):

$$F(x) = \bigoplus_{i \in [m]} h^i \odot x^{a^i} = \min\left\{h^i + \langle a^i, x\rangle \mid i \in [m]\right\}.$$

The tropical polynomial

$$F(x) = \bigoplus_{i\in[m]} h^i \odot x^{a^i} = \min\left\{h^i + \langle a^i, x\rangle \mid i \in [m]\right\}.$$

gives the *tropical hypersurface* defined by F, or *vanishing locus* of F as

$$\mathcal{T}(F) := \left\{x \in \mathbb{R}^d \mid \text{the minimum of } F(x) \text{ is attained at least twice}\right\}.$$

Theorem (Folklore)

There is a bijection between the k-dimensional cells of $\mathcal{S}$ and the $(n-k)$-dimensional cells of $\mathcal{T}(F)$. The bounded cells of $\mathcal{T}(F)$ corresponds to the interior cells of $\mathcal{S}$.

Cayley case

When $\mathcal{A}$ is a Cayley embedding, the tropical phenomena described here can be extended to the mixed subdivision obtained after the Cayley trick.

Theorem (GMPTY, 23')

Let s be a composition and h an admissible height function for the DKK traingulation of $(G_s, \preceq)$. The tropical dual of the Cayley mixed subdivision is the polyhedral complex of cells induced by the arrangement of tropical hypersurfaces

$$\mathcal{H}_{s,h} = \left\{ \mathcal{T}(F_t^j) \,|\, j \in [2, n+1],\, t \in [s_j - 1] \right\},$$

where $F_t^j(x) = \bigoplus_{\delta \in \{0,1\}^{j-1}} -h(R(j,t,\delta)) \odot x^\delta$ and $R(j,t,\delta)$ denotes a route in G_s.

Theorem (GMPTY, 23')

The vertices $v(w)$ of the arrangement $\mathcal{H}_{s,h}$ are in bijection with Stirling s-permutations w and have coordinates

$$v(w)_a = \sum_{t=1}^{s_a} \left(h(l_w^{i(a^t)}) - h(l_w^{i(a^t)+1}) \right).$$

Final realizations

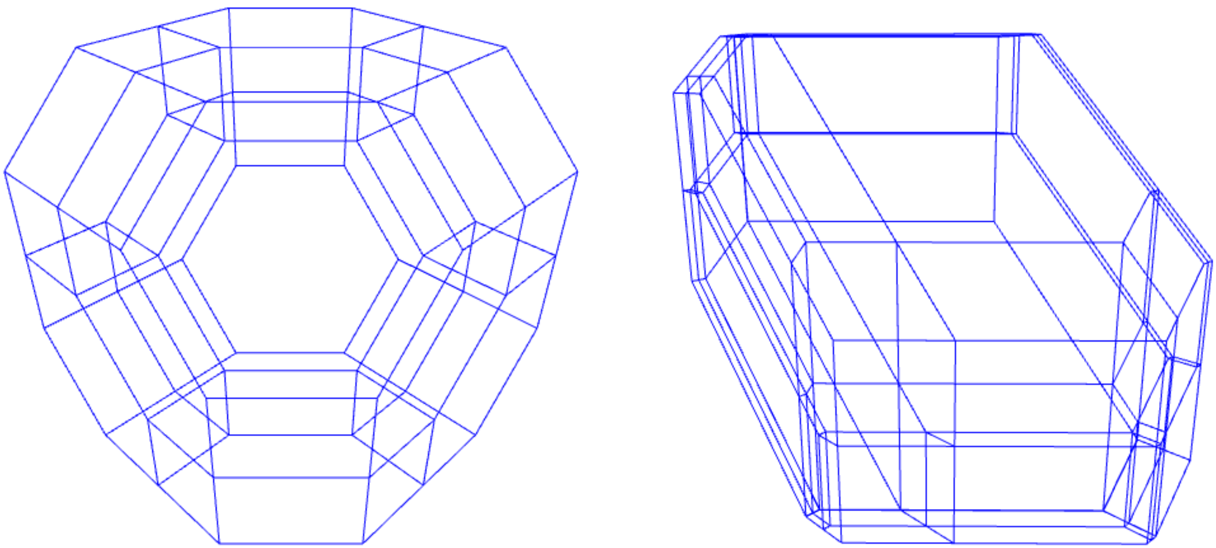


Figure 10: The 1112-permutahedron (left) and the 1222-permutahedron (right) via their tropical realization.

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Thank you!

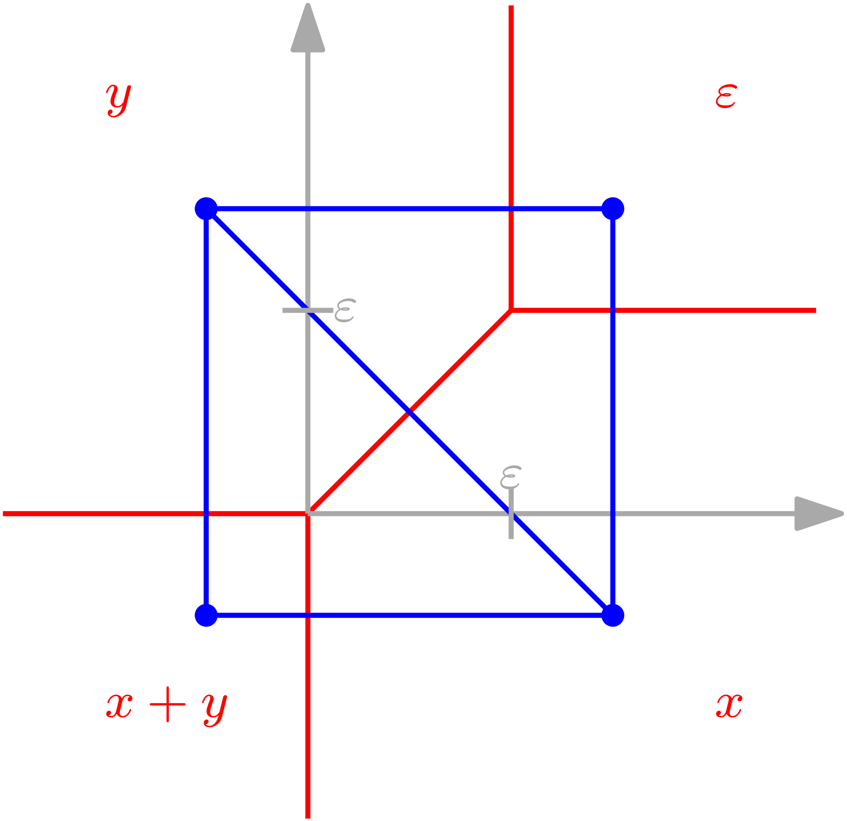
Theorem

Let s be a composition. Let $\varepsilon > 0$ be a small enough real number.
Let h be the function that associates to a route $R = (e^k_{t_k}, \dots, e^1_{t_1})$ of G_S the quantity

$$h(R) = -\sum_{q=1}^{k-1} \varepsilon^q \left(\sum_{j=1}^{k-q} (t_{j+q} + \delta_j)^2 \right),$$

where $\delta_j = \begin{cases} 0 & \text{if } t_j = 0, \\ 1 & \text{if } t_j = s_j \end{cases}$ *for all* $j \in [k-1]$.

Then h is an admissible height function for the DKK triangulation of G_s.


y
ε
ε
ε
$x + y$
x