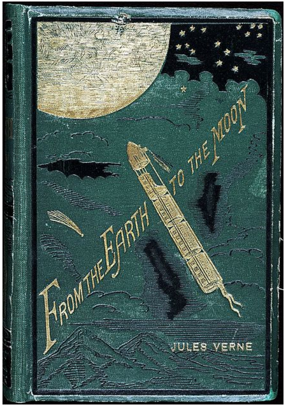
From combinatorics to non-commutative probability: a Hopf-route in 50 minutes

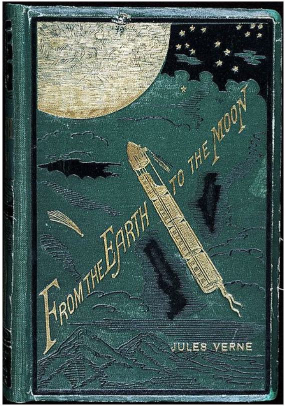
Yannic VARGAS

Beyond Permutahedra and Associahedra: Geometry, Combinatorics, Algebra, and Probability
Weissensee Workshop, 2023



"From the Earth to the Moon: A Direct Route in 97 Hours, 20 Minutes", by Jules Verne, 1865

Purpose of this talk: give a survey of some links between notions in *non-commutative probability*, algebra and combinatorics.



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Some (open and ongoing) problems will be discussed.

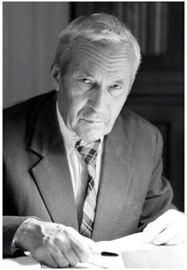
Non-commutative probability space

- The field of *Free Probability* was created by Dan Voiculescu in the 1980s.
- Voiculescu isolated its central concept of *freeness* or, synonymously, *free independence* in the context of operator algebras.
- Philosophy: investigate the notion of "freeness" in analogy to the concept of "independence" from (classical) probability theory (corresponding notions of free independence, free central limit theorem, free convolution...).
- A combinatorial theory of freeness was developed by Nica and Speicher in the 1990s (lattice of non-crossing partitions).
- Voiculescu discovered freeness also asymptotically for many kinds of random matrices (1991).
- Ebrahimi-Fard and Patras studied non-commutative cumulants as infinitesimal characters.

Commutative vs non-commutative probability

D. Voiculescu, R. Speicher, A. Nica

Classical probability space



Andrey Kolmogorov

A **probability space** (Kolmogorov, 1930's) is given by the following data:

- a set Ω (sample space),
- a collection $\mathcal{F}$ (event space),
- $\mathbb{P} : \mathcal{F} \to [0,1]$ (probability function),

satisfying several axioms.

Expectation: for every random variable $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega).$$

Intuition: replace $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ by a more general pair $(\mathcal{A}, \varphi)$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

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Examples: $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, $\left(\mathrm{Mat}_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\right)$, $(\mathrm{Mat}_n(\Omega), \varphi)$.

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$$\varphi(a) := \int_\Omega \mathrm{tr}(a(\omega))\, d\mathbb{P}(\omega)$$

Non-commutative random variable: $a \in \mathcal{A}$

Non-commutative distribution of a:

moments: $(\varphi(a), \varphi(a^2), \varphi(a^3), \dots) \longleftrightarrow \mu_a : \mathbb{C}[x] \to \mathbb{C}, \mu(x^i) := \varphi(a^i)$

Non-commutative join distribution of $\mathbf{a} = (a_1, \dots, a_k)$: if

$1 \leq i_1, \dots, i_n \leq k$,

$$\mu : \mathbb{C}\langle x_1, \dots, x_k \rangle \to \mathbb{C} \quad , \quad \mu_{\mathbf{a}}(x_{i_1} \cdots x_{i_n}) := \varphi(a_{i_1} \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_{i_n})$$

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For example, if $\mathbf{X} = (X_1, \dots, X_n) \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^n$, then

$$\mathbb{E}[p(\mathbf{X})] = \int_{\mathbb{R}^n} p(x) d\mu_{\mathbf{X}}(x), \forall p \in \mathbb{C}[x_1, \dots, x_n],$$

where $\mu_{\mathbf{X}}(B) = \mathbb{P}(\{\omega \in \Omega : \mathbf{X}(\omega) \in B\})$, $B \in \mathcal{B}(\mathbb{R}^n)$.

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In a (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the notion of independence between two random variables $X, Y : \Omega \to \mathbb{C}$ implies

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n).$$

Non-commutative independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$. Let $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that $i_j \neq i_{j+1}$. The family $\{\mathcal{A}_i\}_{i \in I}$ is

- **freely independent** if
$$\varphi(a_1 \cdots a_n) = 0,$$
when $\varphi(a_j) = 0$, for all $1 \leq j \leq n$;
- **boolean independent** if
$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n);$$
- **monotone independent** if
$$\varphi(a_1 \cdots a_n) = \varphi(a_j)\varphi(a_1 \cdots a_{j-1} \cdot a_{j+1} \cdots a_n),$$
when $i_{j-1} < i_j > i_{j+1}$ (I is totally ordered).

Moment to cumulant relations in $(\mathcal{A}, \varphi)$

Consider the multilinear functionals

$$\underset{\text{(Free cumulants)}}{\{r_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}}, \quad \underset{\text{(Boolean cumulants)}}{\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}}, \quad \underset{\text{(Monotone cumulants)}}{\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}}$$

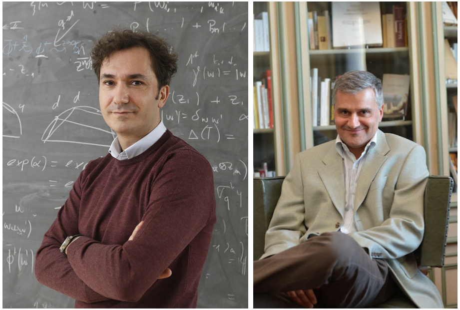
defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} r_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_{\mathrm{Int}}(n)} b_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{\tau(\pi)!} h_\pi(a_1, \ldots, a_n).$$

Algebraic approach to cumulants



Kurusch Ebrahimi-Fard, Frédéric Patras

K. Ebrahimi-Fard and F. Patras proposed an algebraic model to study cumulants in non-commutative probability theory: several (classical and non-commutative) cumulants can be realized as *infinitesimal characters* from a certain Hopf algebra H.

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Algebraic approach to cumulants

- $(\mathcal{A}, \varphi)$ non-commutative probability space.
- $H = T(T_+(\mathcal{A}))$ *words on non-empty words on* $\mathcal{A}$.
- The coproduct Δ in H is *codendriform*: $\Delta = \Delta_< + \Delta_>$.
- The space $(\mathrm{Hom}_{\mathrm{lin}}(H, \mathbb{K}), <, >)$ is a dendriform algebra, with $* =< + >$.
- The linear form φ is extended to $T_+(\mathcal{A})$ by defining to all words $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$

$$\varphi(a_1 a_2 \cdots a_n) := \varphi(a_1 \cdot_{\mathcal{A}} a_2 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n).$$

This is the **multivariate moment** of u.
The map φ is then extended multiplicatively to a map $\Phi : T(T_+(\mathcal{A})) \to \mathbb{K}$ with $\Phi(\mathbf{1}) := 1$ and

$$\Phi(u_1 | \cdots | u_k) := \varphi(u_1) \cdots \varphi(u_k).$$

Cumulants as infinitesimal characters

Proposition (Ebrahimi-Fard, Patras (2015, 2018))

Let $\rho, \kappa, \beta \in \mathfrak{g}(\mathcal{A})$ *the infinitesimal characters solving*

$$\Phi = \exp_*(\rho),$$

$$\Phi = \epsilon + \kappa \prec \Phi$$

and

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Then, ρ, κ, β *correspond to the* ***monotone cumulants, free cumulants and boolean cumulants****, respectively.*

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For any word $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$, we have

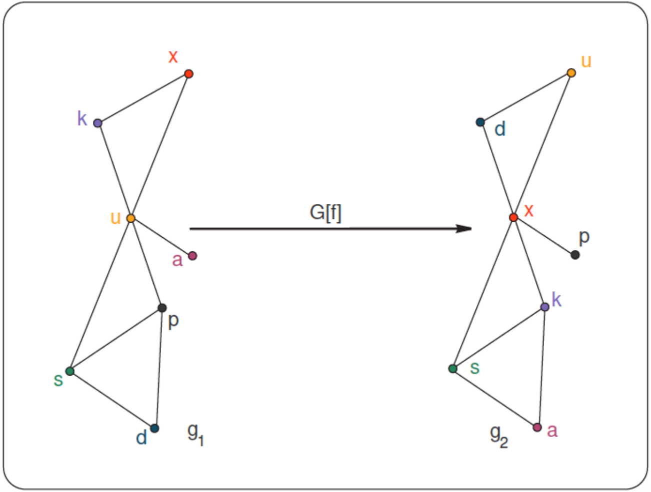
$$h_n(a_1, \dots, a_n) = \rho(u), r_n(a_1, \dots, a_n) = \kappa(u), b_n(a_1, \dots, a_n) = \beta(u).$$

Species

Species

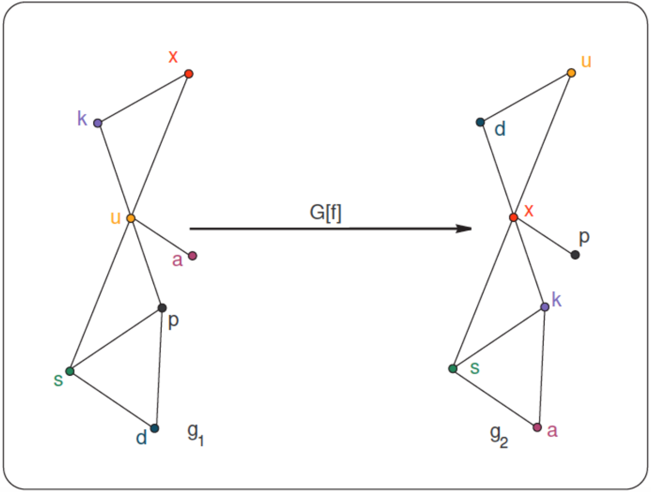
André Joyal, Alain Connes, Olivia Caramello and Laurent Lafforgue, IHES (2015)

The theory of *combinatorial species* was introduced by André Joyal in 1980. Species can be seen as a *categorification* of generating functions. It provides a categorical foundation for enumerative combinatorics.



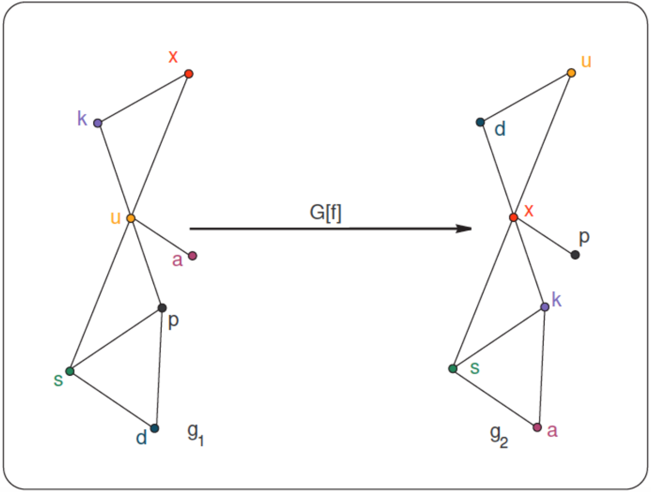
The set of finite graph $G[I]$ with labels in a finite set I satisfies:

(1) If $g_1 \in G[I]$ and $f : I \rightarrow J$ is a bijection, we obtain a graph $g_2 \in G[J]$ after replacing the corresponding labels.



The set of graph $G[I]$ with labels in a finite set I satisfies:

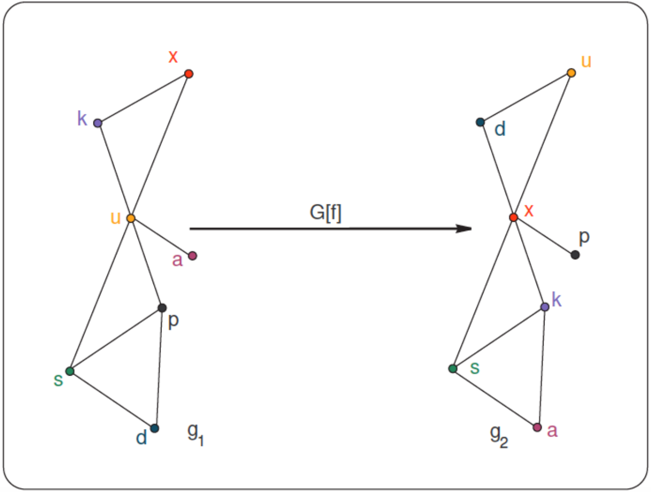
(1) This defines a function $G[f] : G[I] \longrightarrow G[J]$, for every biyection $f : I \longrightarrow J$.



The set of graph in $G[I]$ with labels in a finite set I satisfies:

(2) If $f_1 : I \to J$ and $f_2 : J \to K$ are biyections, then

$$G[f_2 \circ f_1] = G[f_2] \circ G[f_1]$$



The set of graph in G[I] with labels in a finite set I satisfies:

(3) $\mathbf{G}[\mathrm{id}_{\mathrm{I}}] = \mathrm{id}_{\mathbf{G}[\mathrm{I}]}$.

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The **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

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The category of species is symmetric monoidal. We can speak of monoids, comonoids, ..., in species.

$$\mathsf{h}[S] \otimes \mathsf{h}[T] \xrightarrow{\mu_{S,T}} \mathsf{h}[I] \qquad \mathsf{h}[I] \xrightarrow{\Delta_{S,T}} \mathsf{h}[S] \otimes \mathsf{h}[T].$$

Examples of species

- Species E of **sets**:

$$E[I] := \mathbb{K}\{*_I\}.$$

- Species E_n of n-**sets**:

$$E_n[I] := \begin{cases} \mathbb{K}\{*_I\}, & \text{if } |I| = n; \\ (0), & \text{if } |I| \neq n. \end{cases}$$

- Species $X := E_1$ of sets of one element.
- Species Π of **partitions**.
- Species L of **linear orders**.
- Species G of **graphs**:

$$G[I] := \mathbb{K}\{ \text{ finite graphs with vertices in } I \}.$$

Examples of species

- Species B of **binary trees**.
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Dream on...

- Species SM of **simple maps**.
- Species FSM of **fully simple maps**.

Operations on species

- **Sum of species**

$$(p + q)[I] := p[I] \oplus q[I].$$

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- **Product of species** (Cauchy product)

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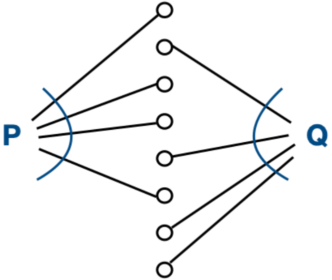
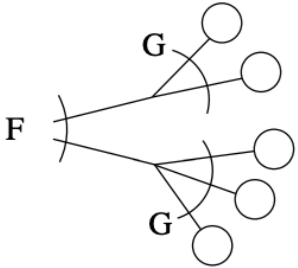


Operations on species

- **Composition of species**

$$(p \circ q)[I] := \bigoplus_{\pi \in \Pi[I]} p[\pi] \otimes \bigotimes_{B \in \pi} q[B].$$



Generating function of a species

To every species p it is associated its **exponential generating function**:

$$p(x) := \sum_{n \geq 0} \dim(p[n]) \frac{x^n}{n!}.$$

We have:

$$(p + q)(x) = p(x) + q(x),$$

$$(p \cdot q)(x) = p(x) \cdot q(x),$$

$$(p \circ q)(x) = p(x) \circ q(x).$$

For the last identity, $q[\emptyset] := (0)$.

A labelled binary tree is:

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This translates as,

$$\mathsf{B} = \mathsf{X} + \mathsf{E}_2 \circ \mathsf{B},$$

which implies:

$$\mathsf{B}(x) = x + \mathsf{B}(x)^2/2.$$

Therefore,

$$\mathsf{B}(x) = 1 - \sqrt{1-2x} = \sum_{n\geq 1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)\frac{x^n}{n!}.$$

Marcelo Aguiar, Swapneel Mahajan

Based on Aguiar, M., Mahajan, S. (2013). *Hopf monoids in the category of species*, Hopf algebras and tensor categories, 585, 17-124.

From species to vector spaces I

There are functors

$$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^{\vee}, \overline{\mathcal{K}} : \text{Hopf monoids in species} \longrightarrow \mathbb{N}\text{-graded Hopf algebras.}$$

$$\mathcal{K}(\mathsf{h}) = \mathcal{K}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]$$

$$\mathcal{K}^{\vee}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]_{\mathfrak{S}_n} \quad , \quad \overline{\mathcal{K}}^{\vee}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]^{\mathfrak{S}_n}$$

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Patras-Schocker-Reutenauer:

$\mathcal{K}(\mathsf{h})$: cosymmetrized bialgebra

$\mathcal{K}^{\vee}(\mathsf{h})$: symmetrized bialgebra

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- $\mathcal{K}(h) \cong \overline{\mathcal{K}}(L \times h)$.
- If h is finite-dimensional, then $\overline{\mathcal{K}}(h^*) \cong \overline{\mathcal{K}}(h)^*$.
- If h is cocommutative, then so are $\mathcal{K}(h)$ and $\overline{\mathcal{K}}(h)$.
- If h is commutative, so is $\overline{\mathcal{K}}(h)$.

From species to vector spaces II

Let p be a species.

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A **series** s of p is a collection of elements

$$s_I \in \mathsf{p}[I],$$

one for each finite set I, such that

$$\mathsf{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

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The space $\mathscr{S}(\mathrm{p})$ of all series of p is a vector space:

$$(s+t)_I = s_I + t_I \quad , \quad (\lambda \cdot s)_I := \lambda\, s_I,$$

for $s, t \in \mathscr{S}(\mathrm{p})$ and $\lambda \in \mathbb{K}$.

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Let E be the exponential map. A series s of p corresponds to the morphism of species

$$\mathrm{E} \longrightarrow \mathrm{p}$$

$$*_I \longmapsto s_I,$$

so $\mathscr{S}(\mathrm{p}) \cong \mathrm{Hom}_{\mathrm{Sp}}(\mathrm{E}, \mathrm{p})$.

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for each bijection $\sigma : I \to J$.

Property (1) implies that each $s_{[n]}$ is an $\mathfrak{S}_n$-invariant element of $\mathrm{p}[n]$. In fact,

$$\mathscr{S}(\mathrm{p}) \cong \prod_{n \geq 0} \mathrm{p}[n]^{\mathfrak{S}_n}$$

$$s \mapsto (s_{[n]})_{n \geq 0}.$$

The functor $\mathscr{S}$

Given a morphism of species $f : p \longrightarrow q$ and a series s of p, define $f(s)$ by

$$f(s)_I := f_I(s_I),$$

for every finite set I. As f commutes with bijections, we have

$$q[\sigma](f(s)_I) = (q[\sigma] \circ f_I)(s_I) = (f_J \circ p[\sigma])(s_I) = f_J(s_J) = f(s)_J,$$

for all bijection $\sigma : I \longrightarrow J$. Then, $f(s)$ is a series of q. This defines a functor

$$\mathscr{S} : \mathsf{Sp} \longrightarrow \mathsf{Vec}.$$

The functor $\mathscr{S}$ is *braided lax monoidal*: it preserves monoids, commutative monoids and Lie monoids.

Decorated series

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A V-**decorated series**, or **decorated series**, is a morphism of species

$$\mathsf{E}_V \longrightarrow \mathsf{p},$$

where E_V is the **exponential decorated exponential** given by

$$\mathsf{E}_V[I] := \mathbb{K}\{f : I \to V\}.$$

Let $\mathscr{S}_V(\mathsf{p})$ be the space of decorated series.

Decorated series

A series s in $\mathscr{S}_V(p)$ is a collection of elements

$$s_{I,f} \in p[I],$$

one for each finite set I and for each map $f : I \to V$, such that

$$p[\sigma](s_{I,f}) = s_{J,f\circ\sigma^{-1}},$$

for each bijection $\sigma : I \to J$.

Cumulants from decorated series (V., 2023)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

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Define $\Phi \in \mathscr{S}_{\mathcal{A}}(\mathsf{P}^*)$ as follows: if I is a finite set and $f : I \to \mathcal{A}$, let

$$\Phi_{I,f} \in \mathsf{P}^*[I]$$

given by

$$\Phi_{I,f}(w_1 w_2 \cdots w_n) := \varphi(w_1) \cdots \varphi(w_n),$$

where for each $w_k = x_1^k \cdots x_r^k \in \mathsf{L}_+[I_k]$,

$$\varphi(w) := (\varphi \circ f)(x_1^k) \cdots (\varphi \circ f)(x_r^k).$$

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Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. For every species p, consider the space $C_{\mathcal{A}}(\mathsf{p}) := \mathscr{S}_{\mathcal{A}}((\mathsf{L} \circ \mathsf{p}_+)^*)$.

- Classical cumulants: $\mathsf{p} = \mathsf{X}$
- Non-commutative cumulants: $\mathsf{p} = \mathsf{L}$

Problem 2 : structure on p giving a more general ripping and sewing coproduct on the *free monoid* $\mathsf{L} \circ \mathsf{p}_+$?

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(In progress: structure of *hereditary species* on p)

- More general notion: $\mathsf{C}_{\mathsf{h}}(\mathsf{p}) := \mathscr{S}(\mathcal{H}(\mathsf{h}, (\mathsf{L} \circ \mathsf{p}_+)^*))$.

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The **cumulants** of h are the integers $k_X(h)$ defined by

$$k_X(h) = \sum_{Y : Y \geq X} \mu(X, Y) \dim_{\mathbb{k}} h(Y).$$

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Classical cumulants from Hopf monoids (Aguiar-Mahajan)

The Möbius function of $\Pi[I]$ satisfies

$$\mu(X, Y) = (-1)^{l(Y)-l(X)} \prod_{B \in X} (n_B - 1)!$$

for $X \leq Y$, where n_B is the number of blocks of Y that refine the block B of X. Therefore,

$$k_X(h) = \prod_{B \in X} k_{|B|}(h),$$

for each partition X of I.

Hopf monoid	Distribution	Moments	Cumulants
L	Exponential of par. 1	$n!$	$(n-1)!$
E	Dirac measure $\delta = 1$	1	$\delta_{n,1}$
Π	Poisson of par. 1	Bell_n	1
Σ	Geometric of par. 1	$\mathrm{OrdBell}_n$	$\sum_{k\geq 1} \frac{k^n}{2^k}$

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Proposition (Aguiar-Mahajan)

For any finite-dimensional cocommutative connected bimonoid h*, the dimension of its primitive part is*

$$\dim_{\Bbbk} \mathcal{P}(h)[I] = k_{|I|}(h).$$

Free and boolean cumulants of h

The **free cumulants** of h are the integers $c_n(\mathsf{h})$ defined by

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Question: are these integers non-negative? What conditions on h?

Generating functions

Given a species h, the *ordinary*, *type* and *exponential* generating functions of h are, respectively,

$$E_h(z) := \sum_{n \geq 0} \dim_{\Bbbk} h[n] \frac{z^n}{n!}, \quad T_h(z) = \sum_{n \geq 0} \dim_{\Bbbk} h[n]_{\mathfrak{S}_n} z^n$$

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No assumptions on cocommutativity. What about $\mathsf{c}_n(\mathsf{h})$?

Questions and work in progress

- An algebraic model for several notions of non-commutative independences was presented by Ebrahimi-Fard and Patras. It involves infinitesimal characters on a certain Hopf algebra.
- Understanding this approach in terms of species and algebraic structures in the monoidal category of species (monoids, comonoids, lie monoids, bimonoids) might give a better insight of the combinatorics behind moment-to-cumulant formale.

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- Other (analytical) notions in non-commutative probability are rich in combinatorics: central limit theorems, infinite divisibility, free entropy,...).
- What's next?

Geometrical notion of independence(s)?

Polytope	Hopf monoid	Independence
Permutahedron	Π	Classical
Associahedron	F	Monotone
Cyclohedron	C	Conditional monotone

Joint work with Cesar Ceballos, Adrián Celestino and Franz Lehner.

Thanks!

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