Generalized Tamari lattices living in flow polytopes

Matias von Bell

Institute for Geometry, TU Graz

PAGCAP Workshop

May 15th, 2023

Overview

Presentation summary:

1. Flow polytopes
2. Framed triangulations
3. Connections to generalized Tamari lattices

Overview

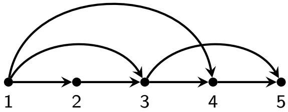
Presentation summary:

1. Flow polytopes
2. Framed triangulations
3. Connections to generalized Tamari lattices

Photo by Krzysztof Niewolny on Unsplash

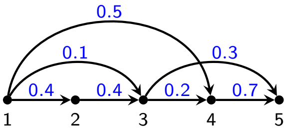
Flow polytopes

A **flow graph** $G = (V, E)$ is a connected acyclic digraph on vertex set $V = \{1, 2, ..., n\}$ with edge multiset E, with edges directed toward larger vertices.



Flow polytopes

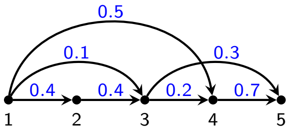
A **flow graph** $G = (V, E)$ is a connected acyclic digraph on vertex set $V = \{1, 2, ..., n\}$ with edge multiset E, with edges directed toward larger vertices.

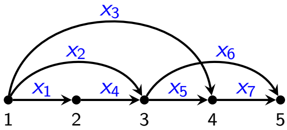


A **flow of size one** on G is a tuple $(x_e)_{e \in E(G)}$ of non-negative real numbers such that flow is preserved at each vertex,

Flow polytopes

A **flow graph** $G = (V, E)$ is a connected acyclic digraph on vertex set $V = \{1, 2, ..., n\}$ with edge multiset E, with edges directed toward larger vertices.



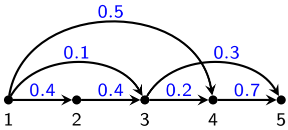


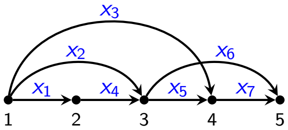
A **flow of size one** on G is a tuple $(x_e)_{e\in E(G)}$ of non-negative real numbers such that flow is preserved at each vertex, i.e.

$$\sum_{e\in \mathrm{Out}(1)} x_e = \sum_{e\in \mathrm{In}(n)} x_e = 1, \text{ and } \sum_{e\in \mathrm{In}(v)} x_e - \sum_{e\in \mathrm{Out}(v)} x_e = 0 \text{ for } v \in [2, n-1].$$

Flow polytopes

A **flow graph** $G = (V, E)$ is a connected acyclic digraph on vertex set $V = \{1, 2, ..., n\}$ with edge multiset E, with edges directed toward larger vertices.





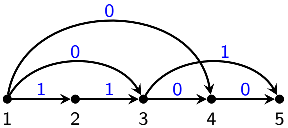
A **flow of size one** on G is a tuple $(x_e)_{e \in E(G)}$ of non-negative real numbers such that flow is preserved at each vertex, i.e.

$$\sum_{e \in \mathrm{Out}(1)} x_e = \sum_{e \in \mathrm{In}(n)} x_e = 1, \text{ and } \sum_{e \in \mathrm{In}(v)} x_e - \sum_{e \in \mathrm{Out}(v)} x_e = 0 \text{ for } v \in [2, n-1].$$

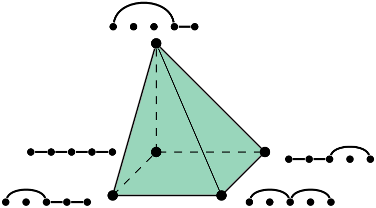
Definition

A **flow polytope** $\mathcal{F}_G$ is the set of all flows of size one in G.

Flow polytopes



G

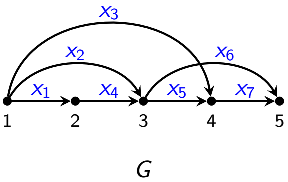


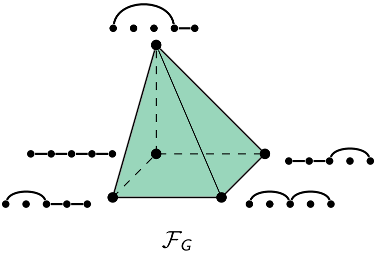
$\mathcal{F}_G$

- Vertices of $\mathcal{F}_G$ correspond with **routes** in G (paths from source to sink).

$$\mathcal{F}_G = \mathrm{conv}\{\mathbf{x}_R \mid R \text{ is a route in } G\}$$

Flow polytopes





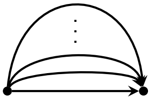
- Vertices of $\mathcal{F}_G$ correspond with **routes** in G (paths from source to sink).

$$\mathcal{F}_G = \mathrm{conv}\{\mathbf{x}_R \mid R \text{ is a route in } G\}$$

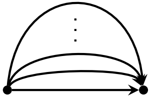
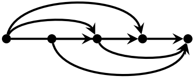
- Dimension:

$$\dim(\mathcal{F}_G) = \sum_{i=1}^{n-1} (|\mathrm{Out}(i)| - 1) = |E(G)| - |V(G)| + 1$$

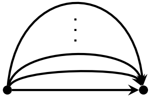
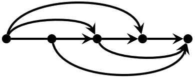
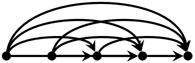
Examples of flow polytopes

Graph	Flow polytope	Normalized Volume
	n-simplex $\mathcal{F}_G = \Delta_n$	1

Examples of flow polytopes

Graph	Flow polytope	Normalized Volume
 $n+1$ edges	n-simplex $\mathcal{F}_G = \Delta_n$	1
 $n+3$ vertices	caracol polytope $\mathcal{F}_{\mathrm{car}(n)}$	$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$

Examples of flow polytopes

Graph	Flow polytope	Normalized Volume
 $n+1$ edges	n-simplex $\mathcal{F}_G = \Delta_n$	1
 $n+3$ vertices	caracol polytope $\mathcal{F}_{\mathrm{car}(n)}$	$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$
 K_{n+3}	Chan–Robbins–Yuen polytope $\mathcal{F}_{K_{n+3}} = \mathrm{CRY}_n$	$\prod_{i=1}^{n} \mathrm{Cat}(i)$ Zeilberger '99 (No combinatorial proof known)

Triangulations

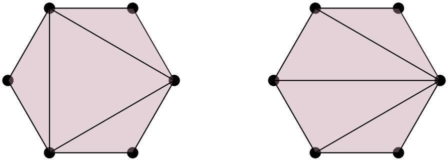
Definition

A **triangulation** $\mathscr{T}$ of a d-dimensional lattice polytope $\mathcal{P}$ is a finite collection of d-simplices $\mathcal{S}_1, \ldots, \mathcal{S}_n$ such that

(i) $\mathcal{P} = \bigcup_{i=1}^{n} \mathcal{S}_i$; and

(ii) $\mathcal{S}_i \cap \mathcal{S}_j$ is a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$ for any pair $\mathcal{S}_i$, $\mathcal{S}_j \in \mathscr{T}$.

A triangulation is **unimodular** if every d-simplex has (normalized) volume 1.



Triangulations

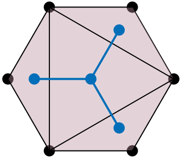
Definition

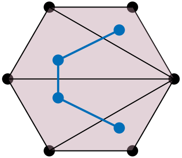
A **triangulation** $\mathscr{T}$ of a d-dimensional lattice polytope $\mathcal{P}$ is a finite collection of d-simplices $\mathcal{S}_1, \ldots, \mathcal{S}_n$ such that

(i) $\mathcal{P} = \bigcup_{i=1}^{n} \mathcal{S}_i$; and

(ii) $\mathcal{S}_i \cap \mathcal{S}_j$ is a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$ for any pair $\mathcal{S}_i$, $\mathcal{S}_j \in \mathscr{T}$.

A triangulation is **unimodular** if every d-simplex has (normalized) volume 1.





Definition

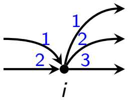
The **dual graph** of a triangulation is the graph on the simplices $\mathcal{S}_1$, ..., $\mathcal{S}_n$, with edges between simplices sharing a common face of codimension one.

Framed triangulations of flow polytopes

Theorem (Danilov–Karzanov–Koshevoy, 2012)

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.

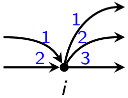


Framed triangulations of flow polytopes

Theorem (Danilov–Karzanov–Koshevoy, 2012)

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.



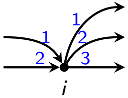
Routes P and Q are **coherent** if Pi and Qi are ordered the same as iP and iQ for any shared vertex i.

Framed triangulations of flow polytopes

Theorem (Danilov–Karzanov–Koshevoy, 2012)

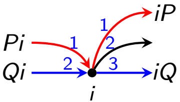
The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.



Routes P and Q are **coherent** if Pi and Qi are ordered the same as iP and iQ for any shared vertex i.

coherent:

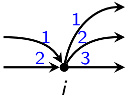


Framed triangulations of flow polytopes

Theorem (Danilov–Karzanov–Koshevoy, 2012)

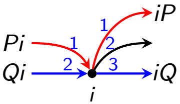
The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.

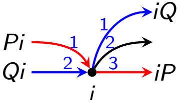


Routes P and Q are **coherent** if Pi and Qi are ordered the same as iP and iQ for any shared vertex i.

coherent:



not coherent:

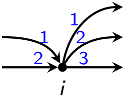


Framed triangulations of flow polytopes

Theorem (Danilov–Karzanov–Koshevoy, 2012)

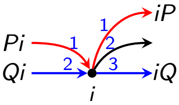
The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

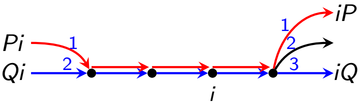
A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.



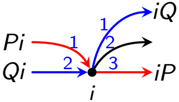
Routes P and Q are **coherent** if Pi and Qi are ordered the same as iP and iQ for any shared vertex i.

coherent:





not coherent:

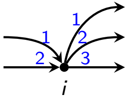


Framed triangulations of flow polytopes

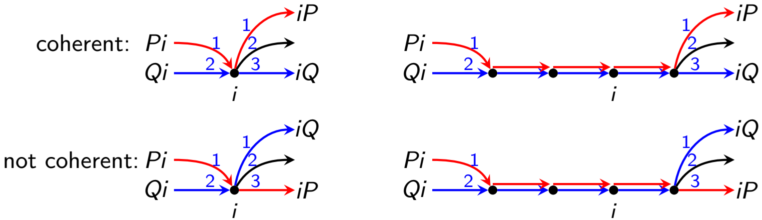
Theorem (Danilov–Karzanov–Koshevoy, 2012)

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

A **framing** is a collection of linear orders on $\mathrm{in}(i)$ and $\mathrm{out}(i)$ for each $i \in V(G)$.



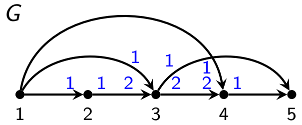
Routes P and Q are **coherent** if Pi and Qi are ordered the same as iP and iQ for any shared vertex i.

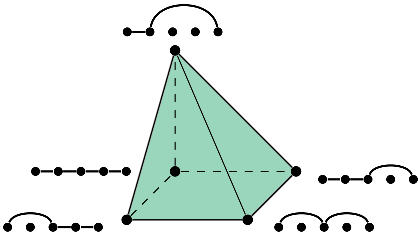


Framed triangulations of flow polytopes

Theorem Danilov–Karzanov–Koshevoy, 2012

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.



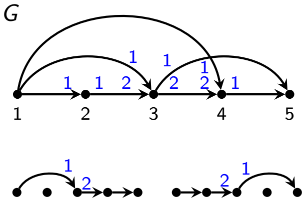


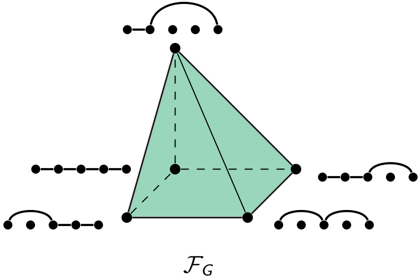
$\mathcal{F}_G$

Framed triangulations of flow polytopes

Theorem Danilov–Karzanov–Koshevoy, 2012

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

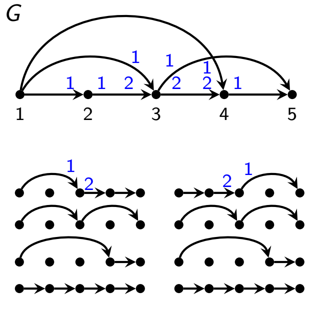


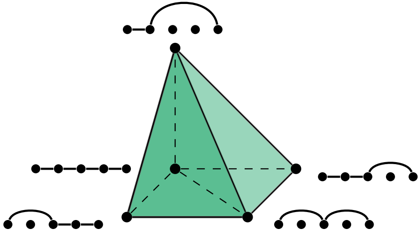


Framed triangulations of flow polytopes

Theorem Danilov–Karzanov–Koshevoy, 2012

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.



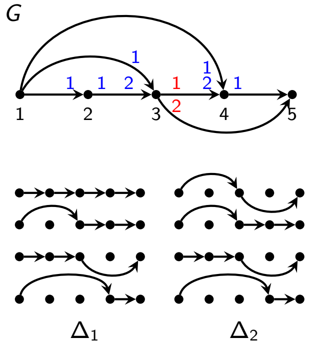


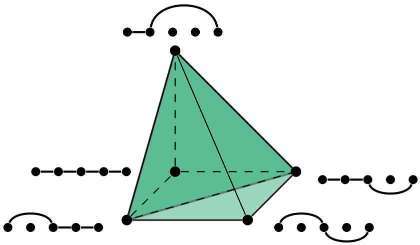
$\mathcal{F}_G$

Framed triangulations of flow polytopes

Theorem Danilov–Karzanov–Koshevoy, 2012

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.



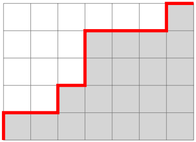


$\mathcal{F}_G$

The ν-caracol flow polytope

Definition

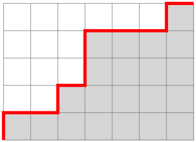
Let ν be a lattice path and let w be the number of valleys in $\overline{\nu} = E\nu N$.
The **ν-caracol graph** $\text{car}(\nu)$ is the graph on $w + 2$ vertices constructed as follows.

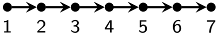
$\nu =$ 

The ν-caracol flow polytope

Definition

Let ν be a lattice path and let w be the number of valleys in $\overline{\nu} = E\nu N$.
The ν**-caracol graph** $\text{car}(\nu)$ is the graph on $w+2$ vertices constructed as follows.

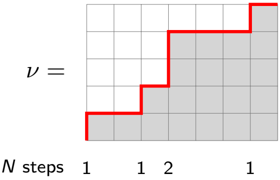
$\nu =$ 

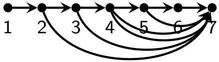


The ν-caracol flow polytope

Definition

Let ν be a lattice path and let w be the number of valleys in $\overline{\nu} = E\nu N$.
The **ν-caracol graph** $\mathrm{car}(\nu)$ is the graph on $w+2$ vertices constructed as follows.

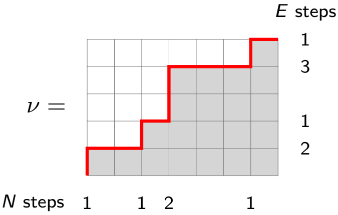
$\nu =$ 

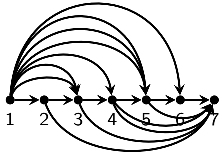


The ν-caracol flow polytope

Definition

Let ν be a lattice path and let w be the number of valleys in $\overline{\nu} = E\nu N$.
The ν**-caracol graph** $\mathrm{car}(\nu)$ is the graph on $w+2$ vertices constructed as follows.

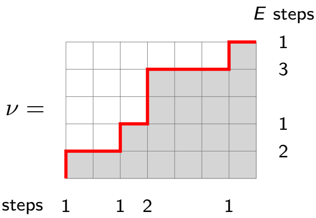


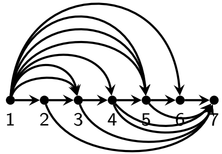


The ν-caracol flow polytope

Definition

Let ν be a lattice path and let w be the number of valleys in $\overline{\nu} = E\nu N$.
The **ν-caracol graph** $\mathrm{car}(\nu)$ is the graph on $w+2$ vertices constructed as follows.





$\mathcal{F}_{\mathrm{car}(\nu)} = $ **ν-caracol flow polytope**

Mészáros–Morales [2019]

$$\mathrm{vol}\,\mathcal{F}_{\mathrm{car}(\nu)} = \mathrm{Cat}(\nu) := \det\left(\binom{1+\sum_{k=1}^{b-j}\nu_k}{1+j-i}\right)_{1\le i,j\le b-1}$$

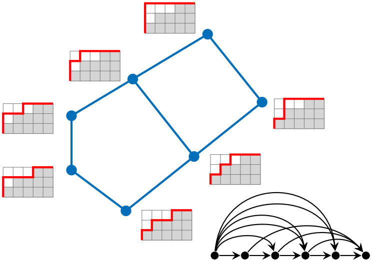
Framed triangulations of $\mathcal{F}_{\text{car}(\nu)}$

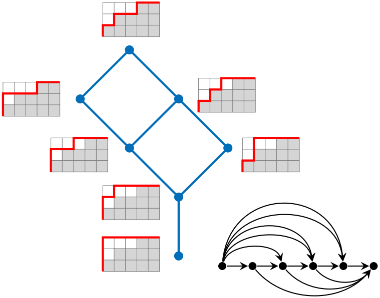
Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

The Hasse diagrams of the following lattices appear as dual graphs in framed triangulations of $\mathcal{F}_{\text{car}(\nu)}$.

(1) *The ν-Tamari lattice.*

(2) *The principal order ideal generated by ν in Young's lattice.*



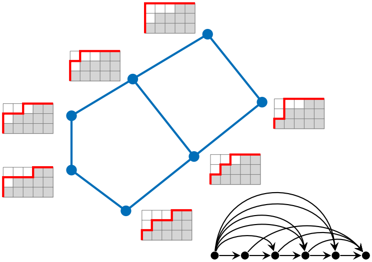


Framed triangulations of $\mathcal{F}_{\mathrm{car}(\nu)}$

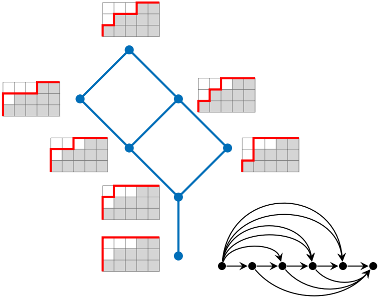
Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

The Hasse diagrams of the following lattices appear as dual graphs in framed triangulations of $\mathcal{F}_{\mathrm{car}(\nu)}$.

1. (1) *The ν-Tamari lattice.*
2. (2) *The principal order ideal generated by ν in Young's lattice.*



A new realization of the ν-Tamari complex of Ceballos–Padrol–Sarmiento



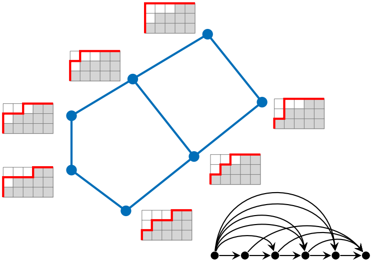
Framed triangulations of $\mathcal{F}_{\mathrm{car}(\nu)}$

Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

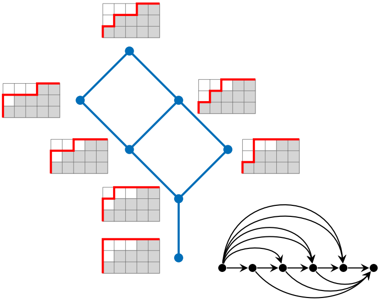
The Hasse diagrams of the following lattices appear as dual graphs in framed triangulations of $\mathcal{F}_{\mathrm{car}(\nu)}$.

(1) *The ν-Tamari lattice.*

(2) *The principal order ideal generated by ν in Young's lattice.*



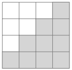
A new realization of the ν-Tamari complex of Ceballos–Padrol–Sarmiento



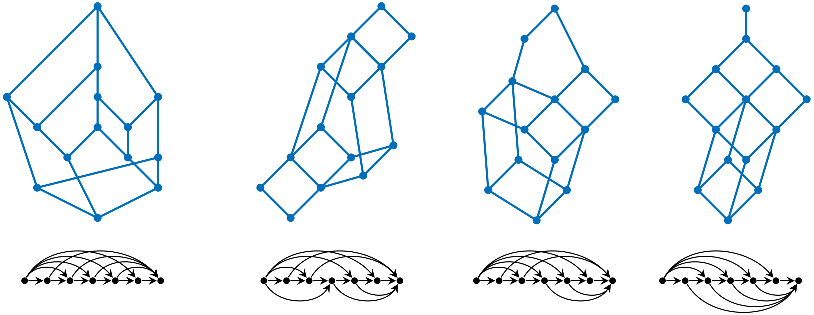
What about other framings?

Other framings?

Current work with Cesar Ceballos

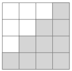


$\nu = NENENENE$

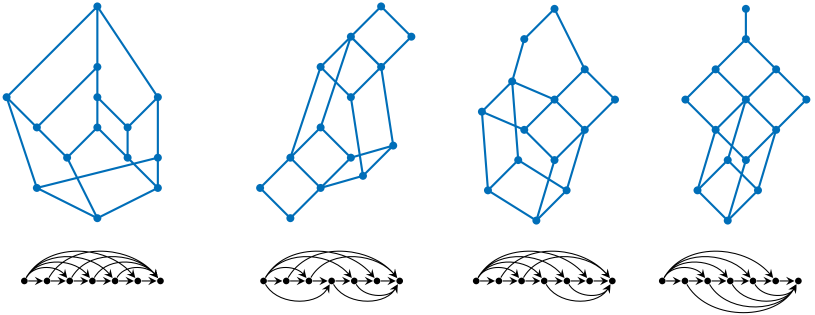


Other framings?

Current work with Cesar Ceballos



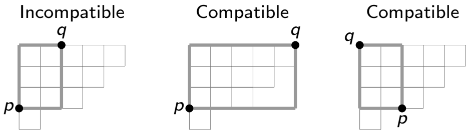
$\nu = NENENENE$



We know for general ν that the dual graphs contain Hasse diagrams of:

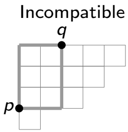
- alt ν-Tamari lattices of Ceballos–Chenevière.
- “cross-Tamari” lattices

Alt ν-Tamari and cross-Tamari lattices

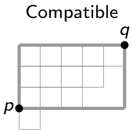


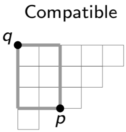
q NE of p

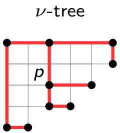
Alt ν-Tamari and cross-Tamari lattices



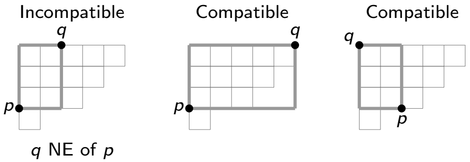
q NE of p

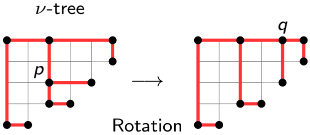




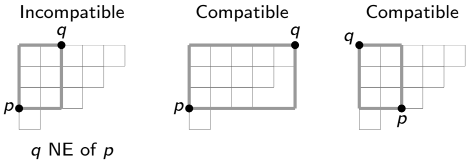


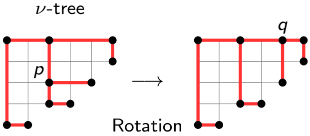
Alt ν-Tamari and cross-Tamari lattices

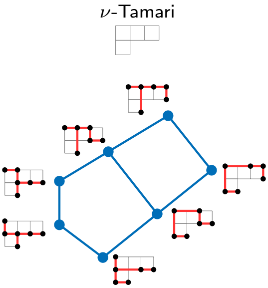




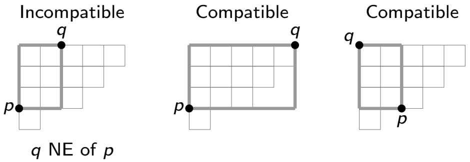
Alt ν-Tamari and cross-Tamari lattices

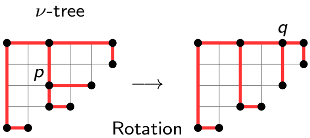


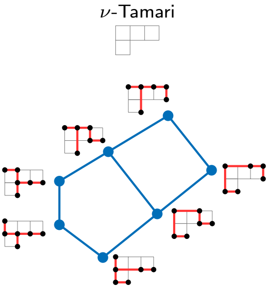


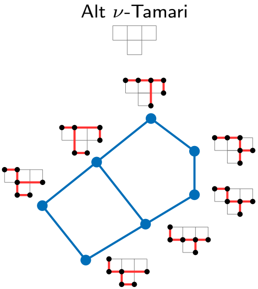


Alt ν-Tamari and cross-Tamari lattices

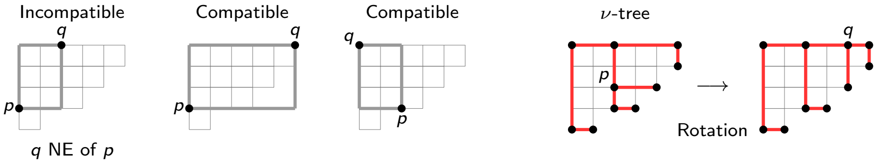


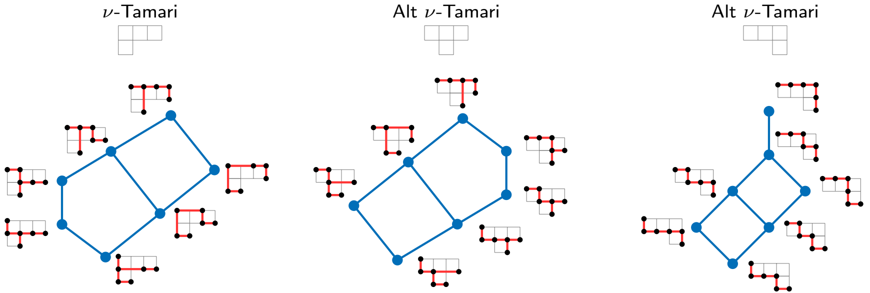






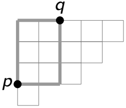
Alt ν-Tamari and cross-Tamari lattices





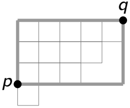
Alt ν-Tamari and cross-Tamari lattices

Incompatible

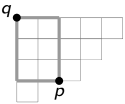


q NE of p

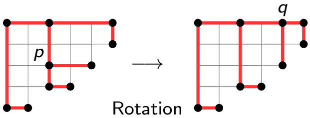
Compatible



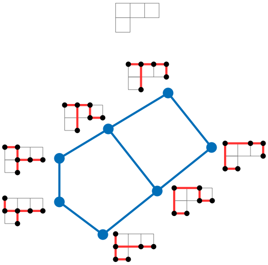
Compatible



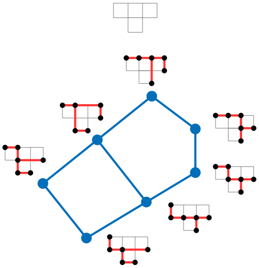
ν-tree



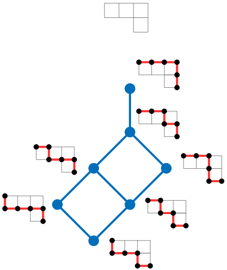
ν-Tamari



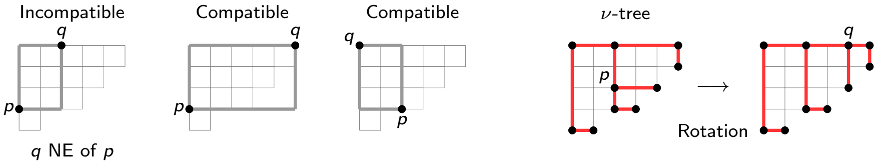
Alt ν-Tamari

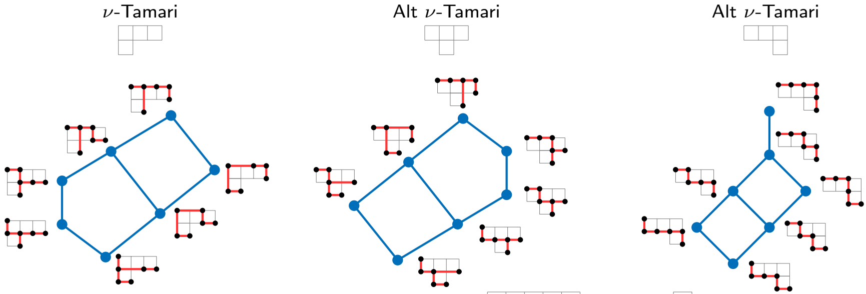


Alt ν-Tamari

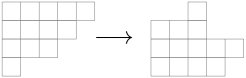


Alt ν-Tamari and cross-Tamari lattices



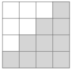


Cross-Tamari: Permute rows and columns!

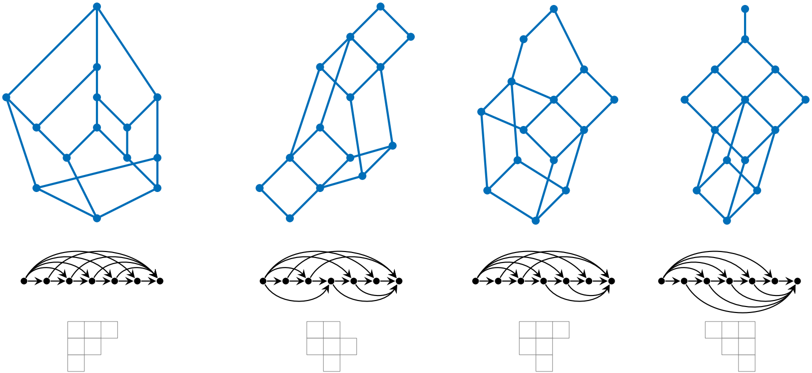


Alt ν-Tamari and cross-Tamari lattices

Current work with Cesar Ceballos

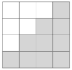


$\nu = NENENENE$

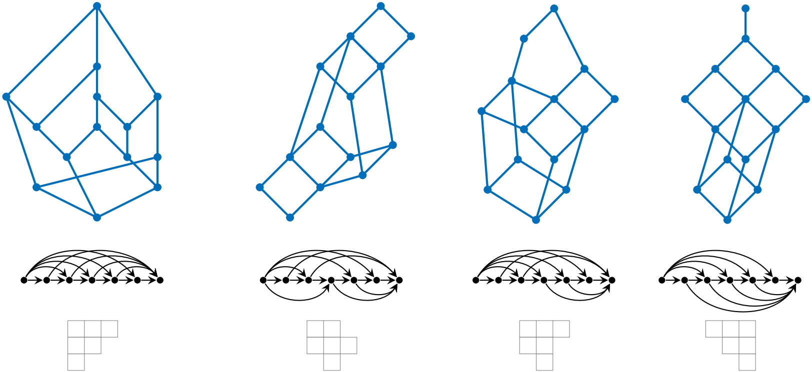


Alt ν-Tamari and cross-Tamari lattices

Current work with Cesar Ceballos

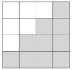


$\nu = NENENENE$

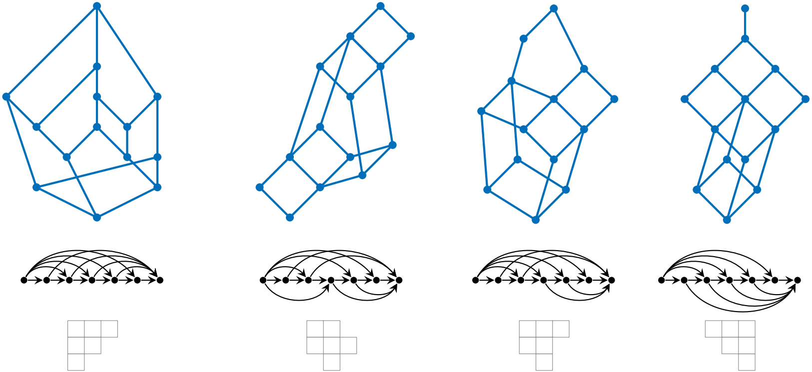


Can the lattice structure be obtained from the framed triangulation?

Current work with Cesar Ceballos



$\nu = NENENENE$



Can the lattice structure be obtained from the framed triangulation?

- Yes! In progress.

Thank you!

1. Matias von Bell, Rafael S. González D'León, Francisco Mayorga Cetina, Martha Yip. "A unifying framework for the ν-Tamari lattice and principal order ideals in Young's lattice". Combinatorica (accepted).
2. Cesar Ceballos, Clement Chenvière. On linear intervals in the alt ν-Tamari lattices. https://arxiv.org/abs/2305.02250
3. Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. "Geometry of ν-Tamari lattices in types A and B". Trans. Amer. Math. Soc., 371(4):2575–2622, 2019.
5. Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. "Coherent fans in the space of flows in framed graphs". In:24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). Discrete Math. Theor.Comput. Sci. Proc., AR. 2012, pp. 481–490.
6. Karola Mészáros and Alejandro H. Morales. "Volumes and Ehrhart polynomials of flow polytopes". In:Math. Z.293.3-4 (2019), pp. 1369–1401.